Silk Road Mathematics Competiton 2016

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by MRF2017, parmenides51

1 Let a, b and c be real numbers such that $|(a-b)(b-c)(c-a)|=1$. Find the smallest value of the expression $|a|+|b|+|c|$. (K.Satylhanov)

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2 Around the acute-angled triangle ABC ($AC>CB$) a circle is circumscribed, and the point N is midpoint of the arc ACB of this circle. Let the points A_1 and B_1 be the feet of perpendiculars on the straight line NC, drawn from points A and B respectively (segment NC lies inside the segment A_1B_1). Altitude A_1A_2 of triangle A_1AC and altitude B_1B_2 of triangle B_1BC intersect at a point K . Prove that $\angle A_1KN = \angle B_1KM$, where M is midpoint of the segment A_2B_2 .

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3 Given natural numbers a, b and function $f:\mathbb{N}\to\mathbb{N}$ such that for any natural number n, $f\left(n+a\right)$ is divided by $f\left(\left[\sqrt{n}\right]+b\right)$. Prove that for any natural n exist n pairwise distinct and pairwise relatively prime natural numbers a_1, a_2, $\ldots$, a_n such that the number $f\left(a_{i+1}\right)$ is divided by $f\left(a_i\right)$ for each $i=1,2,\ldots,n-1$.

(Here $[x]$ is the integer part of number x, that is, the largest integer not exceeding x.)

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4 Let $P(n)$ be the number of ways to split a natural number n to the sum of powers of two, when the order does not matter. For example $P(5)=4$, as $5=4+1=2+2+1=2+1+1+1=1+1+1+1+1$. Prove that for any natural the identity $P(n)+(-1)^{a_1}P(n-1)+(-1)^{a_2}P(n-2)+\ldots+(-1)^{a_{n-1}}P(1)+(-1)^{a_n}=0$, is true, where a_k is the number of units in the binary number record k .

source (`http://matol.kz/comments/2720/show`)
