India International Mathematical Olympiad Training Camp 2017

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– Practice Tests

– Practice Test 1

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1 Let $P_c(x)=x^4+ax^3+bx^2+cx+1$ and $Q_c(x)=x^4+cx^3+bx^2+ax+1$ with a,b real numbers, $c\in\{1,2,\dots,2017\}$ an integer and $a\ne c$. Define $A_c=\{\alpha|P_c(\alpha)=0\}$ and $B_c=\{\beta|P(\beta)=0\}$.

(a) Find the number of unordered pairs of polynomials $P_c(x),Q_c(x)$ with exactly two common roots.

(b) For any $1\le c\le 2017$, find the sum of the elements of $A_c\Delta B_c$.

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2 Find all positive integers $p,q,r,s>1$ such that

$$p!+q!+r!=2^s.$$

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3 Let $ABCD$ be a cyclic quadrilateral inscribed in circle Ω with $AC\perp BD$. Let $P=AC\cap BD$ and W,X,Y,Z be the projections of P on the lines AB,BC,CD,DA respectively. Let E,F,G,H be the mid-points of sides AB,BC,CD,DA respectively.

(a) Prove that E,F,G,H,W,X,Y,Z are concyclic.

(b) If R is the radius of Ω and d is the distance between its centre and P, then find the radius of the circle in (a) in terms of R and d.

– Practice Test 2

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1 In an acute triangle ABC, points D and E lie on side BC with $BD<BE$. Let O_1,O_2,O_3,O_4,O_5,O_6 be the circumcenters of triangles ABD,ADE,AEC,ABE,ADC,ABC, respectively. Prove that O_1,O_3,O_4,O_5 are con-cyclic if and only if A,O_2,O_6 are collinear.

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2 Let a,b,c,d be pairwise distinct positive integers such that

$$\frac{a}{a+b}+\frac{b}{b+c}+\frac{c}{c+d}+\frac{d}{d+a}$$

is an integer. Prove that $a+b+c+d$ is **not** a prime number.

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3 There are n lamps $L_1, L_2, \dots, L_n$ arranged in a circle in that order. At any given time, each lamp is either *on* or *off*. Every second, each lamp undergoes a change according to the following rule:

(a) For each lamp L_i, if L_{i-1}, L_i, L_{i+1} have the same state in the previous second, then L_i is *off* right now. (Indices taken mod n.)

(b) Otherwise, L_i is *on* right now.

Initially, all the lamps are *off*, except for L_1 which is *on*. Prove that for infinitely many integers n all the lamps will be *off* eventually, after a finite amount of time.

– Team Selection Tests

– TST 1

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1 Let a, b, c be distinct positive real numbers with $abc = 1$. Prove that

$$\sum_{\text{cyc}} \frac{a^6}{(a-b)(a-c)} > 15.$$

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2 Define a sequence of integers $a_0 = m, a_1 = n$ and $a_{k+1} = 4a_k - 5a_{k-1}$ for all $k \geq 1$. Suppose $p > 5$ is a prime with $p \equiv 1 \pmod 4$. Prove that it is possible to choose m, n such that $p \nmid a_k$ for any $k \geq 0$.

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3 Let $n \geq 1$ be a positive integer. An $n \times n$ matrix is called *good* if each entry is a non-negative integer, the sum of entries in each row and each column is equal. A *permutation* matrix is an $n \times n$ matrix consisting of n ones and $n(n-1)$ zeroes such that each row and each column has exactly one non-zero entry.

Prove that any *good* matrix is a sum of finitely many *permutation* matrices.

– TST 2

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1 Suppose $f, g \in \mathbb{R}[x]$ are non constant polynomials. Suppose neither of f, g is the square of a real polynomial but $f(g(x))$ is. Prove that $g(f(x))$ is not the square of a real polynomial.

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2 Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

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3 Let $B = (-1, 0)$ and $C = (1, 0)$ be fixed points on the coordinate plane. A nonempty, bounded subset S of the plane is said to be *nice* if

(i) there is a point T in S such that for every point Q in S, the segment TQ lies entirely in S; and

(ii) for any triangle $P_1P_2P_3$, there exists a unique point A in S and a permutation σ of the indices $\{1,2,3\}$ for which triangles ABC and $P_{\sigma(1)}P_{\sigma(2)}P_{\sigma(3)}$ are similar.

Prove that there exist two distinct nice subsets S and S' of the set $\{(x,y) : x \geq 0, y \geq 0\}$ such that if $A \in S$ and $A' \in S'$ are the unique choices of points in (ii), then the product $BA \cdot BA'$ is a constant independent of the triangle $P_1P_2P_3$.

– TST 3

1 Find all positive integers n for which all positive divisors of n can be put into the cells of a rectangular table under the following constraints:

-each cell contains a distinct divisor;
-the sums of all rows are equal; and
-the sums of all columns are equal.

2 Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

3 Prove that for any positive integers a and b we have

$$a+(-1)^b\sum_{m=0}^{a}(-1)^{\lfloor \frac{bm}{a}\rfloor} \equiv b+(-1)^a\sum_{n=0}^{b}(-1)^{\lfloor \frac{an}{b}\rfloor} \pmod 4.$$

– TST 4

1 Let ABC be an acute angled triangle with incenter I. Line perpendicular to BI at I meets BA and BC at points P and Q respectively. Let D, E be the incenters of $\triangle BIA$ and $\triangle BIC$ respectively. Suppose D, P, Q, E lie on a circle. Prove that $AB = BC$.

2 For each $n \geq 2$ define the polynomial

$$f_n(x) = x^n - x^{n-1} - \cdots - x - 1.$$

Prove that

(a) For each $n \geq 2$, $f_n(x) = 0$ has a unique positive real root α_n;

(b) $(\alpha_n)_n$ is a strictly increasing sequence;

(c) $\lim_{n\to\infty} \alpha_n = 2$.

3 Let a be a positive integer which is not a perfect square, and consider the equation

$$k = \frac{x^2 - a}{x^2 - y^2}.$$

Let A be the set of positive integers k for which the equation admits a solution in $\mathbb{Z}^2$ with $x > \sqrt{a}$, and let B be the set of positive integers for which the equation admits a solution in $\mathbb{Z}^2$ with $0 \le x < \sqrt{a}$. Show that $A = B$.