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by goldenturtle, parmenides51, FedeX333X

1 $1.$A bottle in the shape of a cone lies on its base. Water is poured into the bottle until its level reaches a distance of 8 centimeters from the vertex of the cone (measured vertically). We now turn the bottle upside down without changing the amount of water it contains; This leaves an empty space in the upper part of the cone that is 2 centimeters high.

Find the height of the bottle.

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2 Let ABC be an acute-angeled triangle , non-isosceles and with barycentre G (which is , in fact , the intersection of the medians).Let M be the midpoint of BC , and let Ω be the circle with centre G and radius GM , and let N be the point of intersection between Ω and BC that is distinct from M.Let S be the symmetric point of A with respect to N , that is , the point on the line AN such that $AN = NS$. Prove that GS is perpendicular to BC.

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3 Let $x_1, x_2, ..., x_n$ be positive integers,Asumme that in their decimal representations no x_i ”prolongs” x_j.For instance , 123 prolongs 12 , 459 prolongs 4 , but 124 does not prolog 123.
Prove that : $\frac{1}{x_1} + \frac{1}{x_2} + ... + \frac{1}{x_n} < 3.$

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4 $4.$ Let N be an integer greater than 1.Denote by x the smallest positive integer with the following property:there exists a positive integer y strictly less than $x - 1$, such that x divides $N + y$.Prove that x is either p^n or $2p$, where p is a prime number and n is a positive integer

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5 $5.$Let x be a real number with $0 < x < 1$ and let $0.c_1c_2c_3...$ be the decimal expansion of x.Denote by $B(x)$ the set of all subsequences of $c_1c_2c_3$ that consist of 6 consecutive digits.
For instance , $B(\frac{1}{22}) = 045454, 454545, 545454$
Find the minimum number of elements of $B(x)$ as x varies among all irrational numbers with $0 < x < 1$

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6 Let ABC be a triangle with $AB = AC$ and let I be its incenter. Let Γ be the circumcircle of ABC. Lines BI and CI intersect Γ in two new points, M and N respectively. Let D be another point on Γ lying on arc BC not containing A, and let E, F be the intersections of AD with BI and CI, respectively. Let P, Q be the intersections of DM with CI and of DN with BI respectively.

(i) Prove that D, I, P, Q lie on the same circle Ω
(ii) Prove that lines CE and BF intersect on Ω
