Cono Sur Olympiad 2018

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–	Day 1
1	Let $ABCD$ be a convex quadrilateral, where R and S are points in DC and AB, respectively, such that $AD = RC$ and $BC = SA$. Let P, Q and M be the midpoints of RD, BS and CA, respectively. If $\angle MPC + \angle MQA = 90$, prove that $ABCD$ is cyclic.
2	Prove that every positive integer can be formed by the sums of powers of 3, 4 and 7, where do not appear two powers of the same number and with the same exponent. Example: $2 = 7^0 + 7^0$ and $22 = 3^2 + 3^2 + 4^1$ are not valid representations, but $2 = 3^0 + 7^0$ and $22 = 3^2 + 3^0 + 4^1 + 4^0 + 7^1$ are valid representations.
3	Define the product $P_n = 1! \cdot 2! \cdot 3! \cdots (n-1)! \cdot n!$ a) Find all positive integers m, such that $\frac{P_{2020}}{m!}$ is a perfect square. b) Prove that there are infinite many value(s) of n, such that $\frac{P_n}{m!}$ is a perfect square, for at least two positive integers m.
–	Day 2
4	For each interger $n \geq 4$, we consider the m subsets $A_1, A_2, \dots, A_m$ of $\{1, 2, 3, \dots, n\}$, such that A_1 has exactly one element, A_2 has exactly two elements,...., A_m has exactly m elements and none of these subsets is contained in any other set. Find the maximum value of m.
5	Let ABC be an acute-angled triangle with $\angle BAC = 60^\circ$ and with incenter I and circumcenter O. Let H be the point diametrically opposite(antipode) to O in the circumcircle of $\triangle BOC$. Prove that $IH = BI + IC$.
6	A sequence $a_1, a_2, \dots, a_n$ of positive integers is *alagoana*, if for every n positive integer, one have these two conditions I- $a_{n!} = a_1 \cdot a_2 \cdot a_3 \cdots a_n$ II- The number a_n is the n-power of a positive integer. Find all the sequence(s) *alagoana*.