Rioplatense Mathematical Olympiad, Level 3 2001

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–	Day 1
1	Find all integer numbers a, b, m and n, such that the following two equalities are verified: $a^2 + b^2 = 5mn$ and $m^2 + n^2 = 5ab$
2	Let ABC be an acute triangle and A_1, B_1 and C_1, points on the sides BC, CA and AB, respectively, such that $CB_1 = A_1B_1$ and $BC_1 = A_1C_1$. Let D be the symmetric of A_1 with respect to B_1C_1, O and O_1 are the circumcenters of triangles ABC and $A_1B_1C_1$, respectively. If $A \neq D, O \neq O_1$ and AD is perpendicular to OO_1, prove that $AB = AC$.
3	For every integer $n > 1$, the sequence (S_n) is defined by $S_n = \left\lfloor 2^n \underbrace{\sqrt{2+\sqrt{2+...+\sqrt{2}}}}_{n\ radicals} \right\rfloor$ where $\lfloor x \rfloor$ denotes the floor function of x. Prove that $S_{2001} = 2\,S_{2000} + 1$. .
–	Day 2
4	Find all functions $f : R \to R$ such that, for any $x, y \in R$: $f\left(f\left(x\right) - y\right) \cdot f\left(x + f\left(y\right)\right) = x^2 - y^2$
5	Let ABC be a acute-angled triangle with centroid G, the angle bisector of $\angle ABC$ intersects AC in D. Let P and Q be points in BD where $\angle PBA = \angle PAB$ and $\angle QBC = \angle QCB$. Let M be the midpoint of QP, let N be a point in the line GM such that $GN = 2GM$(where G is the segment MN), prove that: $\angle ANC + \angle ABC = 180$
6	For $m = 1, 2, 3, ...$ denote $S(m)$ the sum of the digits of m, and let $f(m) = m + S(m)$. Show that for each positive integer n, there exists a number that appears exactly n times in the sequence $f(1), f(2), ..., f(m), ...$