IMC 2018

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by Tintarn, ThE-dArK-lOrD

– Day 1

1 Let $(a_n)_{n=1}^\infty$ and $(b_n)_{n=1}^\infty$ be two sequences of positive numbers. Show that the following statements are equivalent:

-There is a sequence $(c_n)_{n=1}^\infty$ of positive numbers such that $\sum_{n=1}^\infty \frac{a_n}{c_n}$ and $\sum_{n=1}^\infty \frac{c_n}{b_n}$ both converge;
-$\sum_{n=1}^\infty \sqrt{\frac{a_n}{b_n}}$ converges.

Proposed by Tom Brta, Charles University, Prague

2 Does there exist a field such that its multiplicative group is isomorphism to its additive group?

Proposed by Alexandre Chapovalov, New York University, Abu Dhabi

3 Determine all rational numbers a for which the matrix

$$\begin{pmatrix} a & -a & -1 & 0 \\ a & -a & 0 & -1 \\ 1 & 0 & a & -a \\ 0 & 1 & a & -a \end{pmatrix}$$

is the square of a matrix with all rational entries.

Proposed by Danil Kroes, University of California, San Diego

4 Find all differentiable functions $f:(0,\infty)\to\mathbb{R}$ such that

$$f(b)-f(a)=(b-a)f(\sqrt{ab}) \qquad \text{for all} \qquad a,b>0.$$

Proposed by Orif Ibrogimov, National University of Uzbekistan

5 Let p and q be prime numbers with $p<q$. Suppose that in a convex polygon $P_1,P_2,,P_{pq}$ all angles are equal and the side lengths are distinct positive integers. Prove that

$$P_1P_2+P_2P_3+\cdots+P_kP_{k+1}\geqslant\frac{k^3+k}{2}$$

holds for every integer k with $1\leqslant k\leqslant p$.

Proposed by Ander Lamaison Vidarte, Berlin Mathematical School, Berlin

– Day 2

6 Let k be a positive integer. Find the smallest positive integer n for which there exists k nonzero vectors $v_1, v_2, , v_k$ in $\mathbb{R}^n$ such that for every pair i, j of indices with $|i-j|>1$ the vectors v_i and v_j are orthogonal.

Proposed by Alexey Balitskiy, Moscow Institute of Physics and Technology and M.I.T.

7 Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers such that $a_0=0$ and

$$a_{n+1}^3 = a_n^2 - 8 \quad \text{for} \quad n = 0, 1, 2,$$

Prove that the following series is convergent:

$$\sum_{n=0}^{\infty} |a_{n+1} - a_n|.$$

Proposed by Orif Ibrogimov, National University of Uzbekistan

8 Let $\Omega = \{(x,y,z) \in \mathbb{Z}^3 : y+1 \geqslant x \geqslant y \geqslant z \geqslant 0\}$. A frog moves along the points of Ω by jumps of length 1. For every positive integer n, determine the number of paths the frog can take to reach (n,n,n) starting from $(0,0,0)$ in exactly $3n$ jumps.

Proposed by Fedor Petrov and Anatoly Vershik, St. Petersburg State University

9 Determine all pairs $P(x), Q(x)$ of complex polynomials with leading coefficient 1 such that $P(x)$ divides $Q(x)^2+1$ and $Q(x)$ divides $P(x)^2+1$.

Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro

10 For $R>1$ let $\mathcal{D}_R = \{(a,b) \in \mathbb{Z}^2 : 0 < a^2+b^2 < R\}$. Compute

$$\lim_{R\to\infty} \sum_{(a,b)\in\mathcal{D}_R} \frac{(-1)^{a+b}}{a^2+b^2}.$$

Proposed by Rodrigo Angelo, Princeton University and Matheus Secco, PUC, Rio de Janeiro