Middle European Mathematical Olympiad 2018

www.artofproblemsolving.com/community/c721161

by Tintarn, sqing

– Individual Competition

1 Let Q^+ denote the set of all positive rational number and let $\alpha \in Q^+$. Determine all functions $f : Q^+ \to (\alpha, +\infty)$ satisfying

$$f(\frac{x+y}{\alpha}) = \frac{f(x)+f(y)}{\alpha}$$

for all $x, y \in Q^+$.

2 The two figures depicted below consisting of 6 and 10 unit squares, respectively, are called staircases.
Consider a 2018×2018 board consisting of 2018^2 cells, each being a unit square. Two arbitrary cells were removed from the same row of the board. Prove that the rest of the board cannot be cut (along the cell borders) into staircases (possibly rotated).

3 Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from$A$. Let R and Q be the centroids of triangles ABD and ACD, respectively. Let P be a point on the line segment BC such that $P \neq D$ and points $P\ Q\ R$ and D are concyclic .Prove that the lines $AP\ BQ$ and CR are concurrent.

4 (a) Prove that for every positive integer m there exists an integer $n \geq m$ such that

$$\left\lfloor \frac{n}{1} \right\rfloor \cdot \left\lfloor \frac{n}{2} \right\rfloor \cdots \left\lfloor \frac{n}{m} \right\rfloor = \binom{n}{m} \quad (*)$$

(b) Denote by $p(m)$ the smallest integer $n \geq m$ such that the equation $(*)$ holds. Prove that $p(2018) = p(2019)$.
Remark: For a real number x, we denote by $\lfloor x \rfloor$ the largest integer not larger than x.

– Team Competition

1 Let a, b and c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{a^2-b^2}{a+bc} + \frac{b^2-c^2}{b+ca} + \frac{c^2-a^2}{c+ab} \leq a+b+c-3.$$

2 Let $P(x)$ be a polynomial of degree $n \geq 2$ with rational coefficients such that $P(x)$ has n pairwise different reel roots forming an arithmetic progression .Prove that among the roots of $P(x)$ there are two that are also the roots of some polynomial of degree 2 with rational coefficients .

3 A graup of pirates had an argument and not each of them holds some other two at gunpoint.All the pirates are called one by one in some order.If the called pirate is still alive , he shoots both pirates he is aiming at (some of whom might already be dead .) All shorts are immediatcly lethal . After all the pirates have been called , it turns out the exactly 28 pirates got killed . Prove that if the pirates were called in whatever other order , at least 10 pirates would have been killed anyway.

4 Let n be a positive integer and $u_1, u_2, \cdots, u_n$ be positive integers not larger than 2^k, for some integer $k \geq 3$. A representation of a non-negative integer t is a sequence of non-negative integers $a_1, a_2, \cdots, a_n$ such that $t = a_1u_1 + a_2u_2 + \cdots + a_nu_n$.
Prove that if a non-negative integer t has a representation,then it also has a representation where less than $2k$ of numbers $a_1, a_2, \cdots, a_n$ are non-zero.

5 Let ABC be an acute-angled triangle with $AB < AC$, and let D be the foot of its altitude from$A$, points B' and C' lie on the rays AB and AC, respectively , so that points B', C' and D are collinear and points B, C, B' and C' lie on one circle with center O. Prove that if M is the midpoint of BC and H is the orthocenter of ABC, then $DHMO$ is a parallelogram.

6 Let ABC be a triangle . The internal bisector of ABC intersects the side AC at L and the circumcircle of ABC again at $W \neq B$. Let K be the perpendicular projection of L onto AW. the circumcircle of BLC intersects line CK again at $P \neq C$. Lines BP and AW meet at point T. Prove that

$$AW = WT.$$

7 Let $a_1, a_2, a_3, \cdots$ be the sequence of positive integers such that

$$a_1 = 1, a_{k+1} = a_k^3 + 1,$$

for all positive integers k.
Prove that for every prime number p of the form $3l + 2$, where l is a non-negative integer ,there exists a positive integer n such that a_n is divisible by p.

8 An integer n is called silesian if there exist positive integers a, b and c such that

$$n = \frac{a^2 + b^2 + c^2}{ab + bc + ca}.$$

(a) prove that there are infinitely many silesian integers. (b) prove that not every positive integer is silesian.