National Science Olympiad 2011

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– Day 1

1 For a number n in base 10, let $f(n)$ be the sum of all numbers possible by removing some digits of n (including none and all). For example, if $n = 1234$, $f(n) = 1234 + 123 + 124 + 134 + 234 + 12 + 13 + 14 + 23 + 24 + 34 + 1 + 2 + 3 + 4 = 1979$; this is formed by taking the sums of all numbers obtained when removing no digit from n (1234), removing one digit from n (123, 124, 134, 234), removing two digits from n (12, 13, 14, 23, 24, 34), removing three digits from n (1, 2, 3, 4), and removing all digits from n (0). If p is a 2011-digit integer, prove that $f(p) - p$ is divisible by 9.

Remark: If a number appears twice or more, it is counted as many times as it appears. For example, with the number 101, 1 appears three times (by removing the first digit, giving 01 which is equal to 1, removing the first two digits, or removing the last two digits), so it is counted three times.

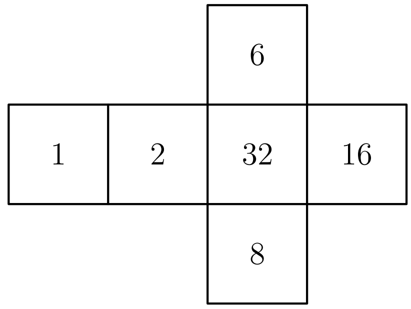
2 For each positive integer n, let s_n be the number of permutations $(a_1, a_2, \cdots, a_n)$ of $(1, 2, \cdots, n)$ such that $\frac{a_1}{1} + \frac{a_2}{2} + \cdots + \frac{a_n}{n}$ is a positive integer. Prove that $s_{2n} \geq n$ for all positive integer n.

3 Given an acute triangle ABC, let l_a be the line passing A and perpendicular to AB, l_b be the line passing B and perpendicular to BC, and l_c be the line passing C and perpendicular to CA. Let D be the intersection of l_b and l_c, E be the intersection of l_c and l_a, and F be the intersection of l_a and l_b. Prove that the area of the triangle DEF is at least three times of the area of ABC.

4 An island has 10 cities, where some of the possible pairs of cities are connected by roads. A *tour route* is a route starting from a city, passing exactly eight out of the other nine cities exactly once each, and returning to the starting city. (In other words, it is a loop that passes only nine cities instead of all ten cities.) For each city, there exists a tour route that doesn't pass the given city. Find the minimum number of roads on the island.

– Day 2

5



The image above is a net of a unit cube. Let n be a positive integer, and let $2n$ such cubes are placed to build a $1 \times 2 \times n$ cuboid which is placed on a floor. Let S be the sum of all numbers on the block visible (not facing the floor). Find the minimum value of n such that there exists such cuboid and its placement on the floor so $S > 2011$.

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6 Let a sequence of integers $a_0, a_1, a_2, \cdots, a_{2010}$ such that $a_0 = 1$ and 2011 divides $a_{k-1}a_k - k$ for all $k = 1, 2, \cdots, 2010$. Prove that 2011 divides $a_{2010} + 1$.

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7 Let $a, b, c \in \mathbb{R}^+$ and $abc = 1$ such that $a^{2011} + b^{2011} + c^{2011} < \frac{1}{a^{2011}} + \frac{1}{b^{2011}} + \frac{1}{c^{2011}}$. Prove that $a + b + c < \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

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8 Given a triangle ABC. Its incircle is tangent to BC, CA, AB at D, E, F respectively. Let K, L be points on CA, AB respectively such that $K \neq A \neq L, \angle EDK = \angle ADE, \angle FDL = \angle ADF$. Prove that the circumcircle of AKL is tangent to the incircle of ABC.
