National Science Olympiad 2015

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by parmenides51, SergeyKrakowska, toedo

– Day 1

1 Albert, Bernard, and Cheryl are playing marbles. At the beginning, each of them brings 5 red marbles, 7 green marbles and 13 blue marbles and in the middle of the table, there is a box of infinitely many red, blue and green marbles. In each turn, each player may choose 2 marbles of different color and replace them with 2 marbles of the third color. After a finite number of steps, this conversation happens.

Albert : " I have only red marbles"
Bernard : "I have only blue marbles"
Cheryl: "I have only green marbles"

Which of the three are lying?

2 For every natural number a and b, define the notation $[a, b]$ as the least common multiple of a and b and the notation (a, b) as the greatest common divisor of a and b. Find all $n \in \mathbb{N}$ that satisfies

$$4\sum_{k=1}^{n}[n,k] = 1 + \sum_{k=1}^{n}(n,k) + 2n^2\sum_{k=1}^{n}\frac{1}{(n,k)}$$

3 Given an acute triangle ABC. Γ_B is a circle that passes through AB, tangent to AC at A and centered at O_B. Define Γ_C and O_C the same way. Let the altitudes of $\triangle ABC$ from B and C meets the circumcircle of $\triangle ABC$ at X and Y, respectively. Prove that A, the midpoint of XY and the midpoint of O_BO_C is collinear.

4 Let function pair $f, g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies

$$f(g(x)y + f(x)) = (y + 2015)f(x)$$

for every $x, y \in \mathbb{R}^+$
a. Prove that $f(x) = 2015g(x)$ for every $x \in \mathbb{R}^+$
b. Give an example of function pair (f, g) that satisfies the statement above and $f(x), g(x) \geq 1$ for every $x \in \mathbb{R}^+$

– Day 2

5 Given positive integers a, b, c, d such that $a \mid c^d$ and $b \mid d^c$. Prove that

$$ab \mid (cd)^{max(a,b)}$$

6 Let ABC be an acute angled triangle with circumcircle O. Line AO intersects the circumcircle of triangle ABC again at point D. Let P be a point on the side BC. Line passing through P perpendicular to AP intersects lines DB and DC at E and F respectively . Line passing through D perpendicular to BC intersects EF at point Q. Prove that $EQ = FQ$ if and only if $BP = CP$.

7 Let a, b, c be positive real numbers. Prove that $\sqrt{\frac{a}{b+c} + \frac{b}{c+a}} + \sqrt{\frac{b}{c+a} + \frac{c}{a+b}} + \sqrt{\frac{c}{a+b} + \frac{a}{b+c}} \geq 3$

8 It is known that there are 3 buildings in the same shape which are located in an equilateral triangle. Each building has a 2015 floor with each floor having one window. In all three buildings, every 1st floor is uninhabited, while each floor of others have exactly one occupant. All windows will be colored with one of red, green or blue. The residents of each floor of a building can see the color of the window in the other buildings of the the same floor and one floor just below it, but they cannot see the colors of the other windows of the two buildings. Besides that, sresidents cannot see the color of the window from any floor in the building itself. For example, resident of the 10th floor can see the colors of the 9th and 10th floor windows for the other buildings (a total of 4 windows) and he can't see the color of the other window. We want to color the windows so that each resident can see at lest 1 window of each color. How many ways are there to color those windows?