National Science Olympiad 2016

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– Day 1

1 Let $ABCD$ be a cyclic quadrilateral wih both diagonals perpendicular to each other and intersecting at point O. Let E, F, G, H be the orthogonal projections of O on sides AB, BC, CD, DA respectively.
a. Prove that $\angle EFG + \angle GHE = 180^o$
b. Prove that OE bisects angle $\angle FEH$.

2 Determine all triples of natural numbers (a, b, c) with $b > 1$ such that $2^c + 2^{2016} = a^b$.

3 There are 5 boxes arranged in a circle. At first, there is one a box containing one ball, while the other boxes are empty. At each step, we can do one of the following two operations:
i. select one box that is not empty, remove one ball from the box and add one ball into both boxes next to the box,
ii. select an empty box next to a non-empty box, from the box the non-empty one moves one ball to the empty box.
Is it possible, that after a few steps, obtained conditions where each box contains exactly $17^{5^{2016}}$ balls?

4 Given triangle ABC such that angles A, B, C satisfy

$$\frac{\cos A}{20} + \frac{\cos B}{21} + \frac{\cos C}{29} = \frac{29}{420}$$

Prove that ABC is right angled triangle

– Day 2

5 Given positive integers a, b, c, d such that $a \mid c^d$ and $b \mid d^c$. Prove that

$$ab \mid (cd)^{max(a,b)}$$

6 For a quadrilateral $ABCD$, we call a square $amazing$ if all of its sides(extended if necessary) pass through distinct vertices of $ABCD$(no side passing through 2 vertices). Prove that for an arbitrary $ABCD$ such that its diagonals are not perpendicular, there exist at least 6 $amazing$ squares

7 Suppose that $p > 2$ is a prime number. For each integer $k = 1, 2, ..., p-1$, denote r_k as the remainder of the division k^p by p^2. Prove that $r_1 + r_2 + r_3 + ... + r_{p-1} = \frac{p^2(p-1)}{2}$

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8 Determine with proof, the number of permutations $a_1, a_2, a_3, ..., a_{2016}$ of $1, 2, 3, ..., 2016$ such that the value of $|a_i - i|$ is fixed for all $i = 1, 2, 3, ..., 2016$, and its value is an integer multiple of 3.
