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– Paper 1

1 Determine, with proof, the smallest positive multiple of 99 all of whose digits are either 1 or 2.

2 Solve the equations :

$$\begin{cases} a+b+c=0 \\ a^2+b^2+c^2=1 \\ a^3+b^3+c^3=4abc \end{cases}$$

for $a, b,$ and c.

3 Four circles are drawn with the sides of quadrilateral $ABCD$ as diameters. The two circles passing through A meet again at A', two circles through B at B' , two circles at C at C' and the two circles at D at D'. Suppose the points A', B', C' and D' are distinct. Prove quadrilateral $A'B'C'D'$ is similar to $ABCD$.

4 An equilateral triangle of integer side length $n \geq 1$ is subdivided into small triangles of unit side length, as illustrated in the figure below for $n = 5$. In this diagram a subtriangle is a triangle of any size which is formed by connecting vertices of the small triangles along the grid lines.
`https://cdn.artofproblemsolving.com/attachments/e/9/17e83ad4872fcf9e97f0479104b9569bf75ac`
`jpg`
It is desired to color each vertex of the small triangles either red or blue in such a way that there is no subtriangle with all of its vertices having the same color. Let $f(n)$ denote the number of distinct colorings satisfying this condition.
Determine, with proof, $f(n)$ for every $n \geq 1$

5 The sequence $a = (a_0, a_1, a_2, ...)$ is defined by $a_0 = 0, a_1 = 2$ and

$$a_{n+2} = 2a_{n+1} + 41a_n$$

Prove that a_{2016} is divisible by 2017.

– Paper 2

1 Does there exist an even positive integer n for which $n+1$ is divisible by 5 and the two numbers $2^n + n$ and $2^n - 1$ are co-prime?

3 A line segment B_0B_n is divided into n equal parts at points $B_1, B_2, ..., B_{n-1}$ and A is a point such that $\angle B_0AB_n$ is a right angle. Prove that :

$$\sum_{k=0}^{n} |AB_k|^2 = \sum_{k=0}^{n} |B_0B_k|^2$$

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2 5 teams play in a soccer competition where each team plays one match against each of the other four teams. A winning team gains 5 points and a losing team 0 points. For a $0-0$ draw both teams gain 1 point, and for other draws ($1-1, 2-2, 3-3$,etc.) both teams gain 2 points. At the end of the competition, we write down the total points for each team, and we find that they form 5 consecutive integers. What is the minimum number of goals scored?

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4 Show that for all non-negative numbers a, b,

$$1 + a^{2017} + b^{2017} \geq a^{10}b^7 + a^7b^{2000} + a^{2000}b^{10}$$

When is equality attained?

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5 Given a positive integer m, a sequence of real numbers $a = (a_1, a_2, a_3, ...)$ is called m-powerful if it satisfies

$$(\sum_{k=1}^{n} a_k)^m = \sum_{k=1}^{n} a_k^m$$

for all positive integers n.
(a) Show that a sequence is 30-powerful if and only if at most one of its terms is non-zero.
(b) Find a sequence none of whose terms are zero but which is 2017-powerful.
