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by parmenides51, tarzanjunior, sqing, MRF2017, basemfouda2002

– Paper 1

1 Given an 8×8 chess board, in how many ways can we select 56 squares on the board while satisfying both of the following requirements:
(a) All black squares are selected.
(b) Exactly seven squares are selected in each column and in each row.

2 Prove that for $N > 1$ that $(N^2)^{2014} - (N^{11})^{106}$ is divisible by $N^6 + N^3 + 1$

3 In the triangle ABC, D is the foot of the altitude from A to BC, and M is the midpoint of the line segment BC. The three angles $\angle BAD$, $\angle DAM$ and $\angle MAC$ are all equal. Find the angles of the triangle ABC.

4 Three different non-zero real numbers a, b, c satisfy the equations $a + \frac{2}{b} = b + \frac{2}{c} = c + \frac{2}{a} = p$, where p is a real number. Prove that $abc + 2p = 0$.

5 Suppose $a_1, a_2, \ldots, a_n > 0$, where $n > 1$ and $\sum_{i=1}^{n} a_i = 1$.
For each $i = 1, 2, \ldots, n$, let $b_i = \frac{a_i^2}{\sum\limits_{j=1}^{n} a_j^2}$. Prove that

$$\sum_{i=1}^{n} \frac{a_i}{1 - a_i} \leq \sum_{i=1}^{n} \frac{b_i}{1 - b_i}.$$

When does equality occur ?

– Paper 2

6 Each of the four positive integers $N, N+1, N+2, N+3$ has exactly six positive divisors. There are exactly20 di erent positive numbers which are exact divisors of at least one of the numbers. One of these is 27. Find all possible values of N.(Both 1 and m are counted as divisors of the number m.)

7 The square $ABCD$ is inscribed in a circle with center O. Let E be the midpoint of AD. The line CE meets the circle again at F. The lines FB and AD meet at H. Prove $HD = 2AH$

8 (a) Let a_0, a_1, a_2 be real numbers and consider the polynomial $P(x) = a_0 + a_1 x + a_2 x^2$.
Assume that $P(-1), P(0)$ and $P(1)$ are integers.

Prove that $P(n)$ is an integer for all integers n.
(b) Let a_0, a_1, a_2, a_3 be real numbers and consider the polynomial $Q(x) = a0 + a_1x + a_2x^2 + a_3x^3$. Assume that there exists an integer i such that $Q(i), Q(i+1), Q(i+2)$ and $Q(i+3)$ are integers. Prove that $Q(n)$ is an integer for all integers n.

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9 Let n be a positive integer and $a_1, ..., a_n$ be positive real numbers.
Let $g(x)$ denote the product $(x + a_1) \cdot ... \cdot (x + a_n)$.
Let a_0 be a real number and let $f(x) = (x - a_0)g(x) = x^{n+1} + b_1x^n + b_2x^{n-1} + ... + b_nx + b_{n+1}$.
Prove that all the coeffcients $b_1, b_2, ..., b_{n+1}$ of the polynomial $f(x)$ are negative if and only if $a_0 > a_1 + a_2 + ... + a_n$.

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10 Over a period of k consecutive days, a total of 2014 babies were born in a certain city, with at least one baby being born each day. Show that:
(a) If $1014 < k \le 2014$, there must be a period of consecutive days during which exactly 100 babies were born.
(b) By contrast, if $k = 1014$, such a period might not exist.
