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– Paper 1

1 In the triangle ABC, the length of the altitude from A to BC is equal to 1. D is the midpoint of AC. What are the possible lengths of BD?

2 A regular polygon with $n \geq 3$ sides is given. Each vertex is coloured either red, green or blue, and no two adjacent vertices of the polygon are the same colour. There is at least one vertex of each colour.
Prove that it is possible to draw certain diagonals of the polygon in such a way that they intersect only at the vertices of the polygon and they divide the polygon into triangles so that each such triangle has vertices of three different colours.

3 Find all positive integers n for which both $837 + n$ and $837 - n$ are cubes of positive integers.

4 Two circles C_1 and C_2, with centres at D and E respectively, touch at B. The circle having DE as diameter intersects the circle C_1 at H and the circle C_2 at K. The points H and K both lie on the same side of the line DE. HK extended in both directions meets the circle C_1 at L and meets the circle C_2 at M. Prove that
(a) $|LH| = |KM|$
(b) the line through B perpendicular to DE bisects HK.

5 Suppose a doubly infinite sequence of real numbers $..., a_{-2}, a_{-1}, a_0, a_1, a_2, ...$ has the property that

$$a_{n+3} = \frac{a_n + a_{n+1} + a_{n+2}}{3},$$

for all integers n. Show that if this sequence is bounded (i.e., if there exists a number R such that $|a_n| \leq R$ for all n), then a_n has the same value for all n.

– Paper 2

6 Suppose x, y are nonnegative real numbers such that $x + y \leq 1$. Prove that $8xy \leq 5x(1 - x) + 5y(1 - y)$
and determine the cases of equality.

7 Let $n > 1$ be an integer and $\Omega = \{1, 2, ..., 2n - 1, 2n\}$ the set of all positive integers that are not larger than $2n$.
A nonempty subset S of Ω is called *sum-free* if, for all elements x, y belonging to S, $x + y$ does

not belong to S. We allow $x = y$ in this condition.
Prove that Ω has more than 2^n distinct *sum-free* subsets.

8 In triangle $\triangle ABC$, the angle $\angle BAC$ is less than 90^o. The perpendiculars from C on AB and from B on AC intersect the circumcircle of $\triangle ABC$ again at D and E respectively. If $|DE| = |BC|$, find the measure of the angle $\angle BAC$.

9 Let $p(x)$ and $q(x)$ be non-constant polynomial functions with integer coeffcients. It is known that the polynomial $p(x)q(x) - 2015$ has at least 33 different integer roots. Prove that neither $p(x)$ nor $q(x)$ can be a polynomial of degree less than three.

10 Prove that, for all pairs of nonnegative integers, j, n,

$$\sum_{K=0}^{n} k^j \binom{n}{k} \geq 2^{n-j} n^j$$