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– Paper 1

1 Mary and Pat play the following number game. Mary picks an initial integer greater than 2017. She then multiplies this number by 2017 and adds 2 to the result. Pat will add 2019 to this new number and it will again be Marys turn. Both players will continue to take alternating turns. Mary will always multiply the current number by 2017 and add 2 to the result when it is her turn. Pat will always add 2019 to the current number when it is his turn. Pat wins if any of the numbers obtained by either player is divisible by 2018. Mary wants to prevent Pat from winning the game.
Determine, with proof, the smallest initial integer Mary could choose in order to achieve this.

2 The triangle ABC is right-angled at A. Its incentre is I, and H is the foot of the perpendicular from I on AB. The perpendicular from H on BC meets BC at E, and it meets the bisector of $\angle ABC$ at D. The perpendicular from A on BC meets BC at F. Prove that $\angle EFD = 45^o$

3 Find all functions $f(x) = ax^2 + bx + c$, with $a \neq 0$, such that $f(f(1)) = f(f(0)) = f(f(-1))$.

4 We say that a rectangle with side lengths a and b *fits inside* a rectangle with side lengths c and d if either ($a \leq c$ and $b \leq d$) or ($a \leq d$ and $b \leq c$). For instance, a rectangle with side lengths 1 and 5 *fits inside* another rectangle with side lengths 1 and 5, and also *fits inside* a rectangle with side lengths 6 and 2.
Suppose S is a set of 2019 rectangles, all with integer side lengths between 1 and 2018 inclusive. Show that there are three rectangles A, B, and C in S such that A fits inside B, and B *fits inside* C.

5 Points A, B and P lie on the circumference of a circle Ω_1 such that $\angle APB$ is an obtuse angle. Let Q be the foot of the perpendicular from P on AB. A second circle Ω_2 is drawn with centre P and radius PQ. The tangents from A and B to Ω_2 intersect Ω_1 at F and H respectively. Prove that FH is tangent to Ω_2.

– Paper 2

6 Find all real-valued functions f satisfying $f(2x+f(y))+f(f(y)) = 4x+8y$ for all real numbers x and y.

7 Let a, b, c be the side lengths of a triangle. Prove that $2(a^3+b^3+c^3) < (a+b+c)(a^2+b^2+c^2) \leq 3(a^3+b^3+c^3)$

8   Let M be the midpoint of side BC of an equilateral triangle ABC. The point D is on CA extended such that A is between D and C. The point E is on AB extended such that B is between A and E, and $|MD| = |ME|$. The point F is the intersection of MD and AB. Prove that $\angle BFM = \angle BME$.

9   The sequence of positive integers $a_1, a_2, a_3, \ldots$ satisfies $a_{n+1} = a_n^2 + 2018$ for $n \geq 1$.
Prove that there exists at most one n for which a_n is the cube of an integer.

10   The game of Greed starts with an initial configuration of one or more piles of stones.
Player 1 and Player 2 take turns to remove stones, beginning with Player 1. At each turn, a player has two choices:
take one stone from any one of the piles (a simple move);
take one stone from each of the remaining piles (a greedy move).
The player who takes the last stone wins.
Consider the following two initial configurations:
(a) There are 2018 piles, with either 20 or 18 stones in each pile.
(b) There are four piles, with $17, 18, 19$, and 20 stones, respectively.
In each case, find an appropriate strategy that guarantees victory to one of the players.