Regional Olympiad - Federation of Bosnia and Herzegovina 2018

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–	Sarajevo, March 24th
–	Grade 9
1	if a, b and c are real numbers such that $(a-b)(b-c)(c-a) \neq 0$, prove the equality: $\frac{b^2c^2}{(a-b)(a-c)}+\frac{c^2a^2}{(b-c)(b-a)}+\frac{a^2b^2}{(c-a)(c-b)}=ab+bc+ca$
2	Determine all triplets (a,b,c) of real numbers such that sets $\{a^2-4c, b^2-2a, c^2-2b\}$ and $\{a-c, b-4c, a+b\}$ are equal and $2a+2b+6=5c$. In every set all elements are pairwise distinct
3	Let p and q be prime numbers such that p^2+pq+q^2 is perfect square. Prove that p^2-pq+q^2 is prime
4	Prove that among arbitrary 13 points in plane with coordinates as integers, such that no three are collinear, we can pick three points as vertices of triangle such that its centroid has coordinates as integers.
5	Let H be an orhocenter of an acute triangle ABC and M midpoint of side BC. If D and E are foots of perpendicular of H on internal and external angle bisector of angle $\angle BAC$, prove that M, D and E are collinear
–	Grade 10
1	Show that system of equations $2ab=6(a+b)-13$ $a^2+b^2=4$ has not solutions in set of real numbers.
2	Find all positive integers n such that number $n^4-4n^3+22n^2-36n+18$ is perfect square of positive integer
3	Solve equation $x\lfloor x \rfloor + \{x\}=2018$, where x is real number
4	Let P be a point on circumcircle of triangle ABC on arc $\overset{\frown}{BC}$ which does not contain point A. Let lines AB and CP intersect at point E, and lines AC and BP intersect at F. If perpendicular bisector of side AB intersects AC in point K, and perpendicular bisector of side AC intersects side AB in point J, prove that: $\left(\frac{CE}{BF}\right)^2=\frac{AJ\cdot JE}{AK\cdot KF}$

5 Board with dimesions 2018×2018 is divided in unit cells 1×1. In some cells of board are placed black chips and in some white chips (in every cell maximum is one chip). Firstly we remove all black chips from columns which contain white chips, and then we remove all white chips from rows which contain black chips. If W is number of remaining white chips, and B number of remaining black chips on board and $A = min\{W, B\}$, determine maximum of A

– Grade 11

1 Find all values of real parameter a for which equation $2\sin^4(x) + \cos^4(x) = a$ has real solutions

2 Let $a_1, a_2, ..., a_{2018}$ be a sequence of numbers such that all its elements are elements of a set $\{-1, 1\}$. Sum

$$S = \sum_{1 \leq i < j \leq 2018} a_i a_j$$

can be negative and can also be positive. Find the minimal value of this sum

3 In triangle ABC given is point P such that $\angle ACP = \angle ABP = 10^\circ$, $\angle CAP = 20^\circ$ and $\angle BAP = 30^\circ$. Prove that $AC = BC$

4 We observe that number $10001 = 73 \cdot 137$ is not prime. Show that every member of infinite sequence $10001, 100010001, 1000100010001, ...$ is not prime

5 It is given 2018 points in plane. Prove that it is possible to cover them with circles such that: $i)$ sum of lengths of all diameters of all circles is not greater than 2018 $ii)$ distance between any two circles is greater than 1

– Grade 12

1 $a)$ Prove that for all positive integers $n \geq 3$ holds:

$$\binom{n}{1} + \binom{n}{2} + ... + \binom{n}{n-1} = 2^n - 2$$

where $\binom{n}{k}$, with integer k such that $n \geq k \geq 0$, is binomial coefficent

$b)$ Let $n \geq 3$ be an odd positive integer. Prove that set $A = \left\{ \binom{n}{1}, \binom{n}{2}, ..., \binom{n}{\frac{n-1}{2}} \right\}$ has odd number of odd numbers

2 Problem 4 from grade 11

3 If numbers x_1, x_2,...,x_n are from interval $\left(\frac{1}{4}, 1\right)$ prove the inequality: $\log_{x_1}\left(x_2 - \frac{1}{4}\right) + \log_{x_2}\left(x_3 - \frac{1}{4}\right) + ... + \log_{x_{n-1}}\left(x_n - \frac{1}{4}\right) + \log_{x_n}\left(x_1 - \frac{1}{4}\right) \geq 2n$

4 Let $ABCD$ be a cyclic quadrilateral and let k_1 and k_2 be circles inscribed in triangles ABC and ABD. Prove that external common tangent of those circles (different from AB) is parallel with CD

5 Problem 5 from grade 11