Bosnia and Herzegovina Team Selection Test 1997

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by gobathegreat

–	Day 1

1 Solve system of equation

$$8(x^3+y^3+z^3)=73$$
$$2(x^2+y^2+z^2)=3(xy+yz+zx)$$
$$xyz=1$$

in set $\mathbb{R}^3$

2 In isosceles triangle ABC with base side AB, on side BC it is given point M. Let O be a circumcenter and S incenter of triangle ABC. Prove that

$$SM \mid\mid AC \Leftrightarrow OM \perp BS$$

3 It is given function $f : A \rightarrow \mathbb{R}$, $(A \subseteq \mathbb{R})$ such that

$$f(x+y)=f(x) \cdot f(y) - f(xy) + 1; (\forall x,y \in A)$$

If $f : A \rightarrow \mathbb{R}$, $(\mathbb{N} \subseteq A \subseteq \mathbb{R})$ is solution of given functional equation, prove that:

$$f(n)=\begin{cases} \frac{c^{n+1}-1}{c-1}, \forall n \in \mathbb{N}, c \neq 1 \\ n+1, \forall n \in \mathbb{N}, c=1 \end{cases}$$

where $c=f(1)-1$ $a)$ Solve given functional equation for $A=\mathbb{N}$ $b)$ With $A=\mathbb{Q}$, find all functions f which are solutions of the given functional equation and also $f(1997) \neq f(1998)$

–	Day 2

4 $a)$ In triangle ABC let A_1, B_1 and C_1 be touching points of incircle ABC with BA, CA and AB, respectively. Let l_1, l_2 and l_3 be lenghts of arcs B_1C_1, A_1C_1, B_1A_1 of incircle ABC, respectively, which does not contain points A_1, B_1 and C_1, respectively.
Does the following inequality hold:

$$\frac{a}{l_1}+\frac{b}{l_2}+\frac{c}{l_3} \geq \frac{9\sqrt{3}}{\pi}$$

$b)$ Tetrahedron $ABCD$ has three pairs of equal opposing sides. Find length of height of tetrahedron in function od lengths of sides

5 $a)$ Prove that for all positive integers n exists a set M_n of positive integers with exactly n elements and:

$i)$ Arithmetic mean of arbitrary non-empty subset of M_n is integer $ii)$ Geometric mean of arbitrary non-empty subset of M_n is integer $iii)$ Both arithmetic mean and geometry mean of arbitrary non-empty subset of M_n is integer

$b)$ Does there exist infinite set M of positive integers such that arithmetic mean of arbitrary non-empty subset of M is integer

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6 Let k, m and n be integers such that $1 < n \leq m-1 \leq k$. Find maximum size of subset S of set $\{1,2,...,k\}$ such that sum of any n different elements from S is not: $a)$ equal to m, $b)$ exceeding m

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