Bosnia and Herzegovina Team Selection Test 1999

www.artofproblemsolving.com/community/c733393

by gobathegreat

– Day 1

1 Let a, b and c be lengths of sides of triangle ABC. Prove that at least one of the equations

$$x^2-2bx+2ac=0$$

$$x^2-2cx+2ab=0$$

$$x^2-2ax+2bc=0$$

does not have real solutions

2 Prove the inequality

$$\frac{a^2}{b+c-a}+\frac{b^2}{a+c-b}+\frac{c^2}{a+b-c}\geq 3\sqrt{3}R$$

in triangle ABC where a, b and c are sides of triangle and R radius of circumcircle of ABC

3 Let $f:[0,1]\rightarrow \mathbb{R}$ be injective function such that $f(0)+f(1)=1$. Prove that exists $x_1,x_2 \in [0,1]$, $x_1 \neq x_2$ such that $2f(x_1)<f(x_2)+\frac{1}{2}$. After that state at least one generalization of this result

– Day 2

4 Let angle bisectors of angles $\angle BAC$ and $\angle ABC$ of triangle ABC intersect sides BC and AC in points D and E, respectively. Let points F and G be foots of perpendiculars from point C on lines AD and BE, respectively. Prove that $FG \mid\mid AB$

5 For any nonempty set S, we define $\sigma(S)$ and $\pi(S)$ as sum and product of all elements from set S, respectively. Prove that $a)$ $\sum \frac{1}{\pi(S)}=n$ $b)$ $\sum \frac{\sigma(S)}{\pi(S)}=(n^2+2n)-\left(1+\frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}\right)(n+1)$ where $\sum$ denotes sum by all nonempty subsets S of set $\{1,2,...,n\}$

6 It is given polynomial

$$P(x)=x^4+3x^3+3x+p, (p \in \mathbb{R})$$

$a)$ Find p such that there exists polynomial with imaginary root x_1 such that $\mid x_1 \mid =1$ and $2Re(x_1)=\frac{1}{2}\left(\sqrt{17}-3\right)$ $b)$ Find all other roots of polynomial P $c)$ Prove that does not exist positive integer n such that $x_1^n=1$