Regional Olympiad - Federation of Bosnia and Herzegovina 2016

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by gobathegreat

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– Grade 9

1 Find minimal value of $A=\frac{\left(x+\frac{1}{x}\right)^6-\left(x^6+\frac{1}{x^6}\right)-2}{\left(x+\frac{1}{x}\right)^3+\left(x^3+\frac{1}{x^3}\right)}$

2 Let ABC be an isosceles triangle such that $\angle BAC=100^{\circ}$. Let D be an intersection point of angle bisector of $\angle ABC$ and side AC, prove that $AD+DB=BC$

3 Nine lines are given such that every one of them intersects given square $ABCD$ on two trapezoids, which area ratio is $2:3$. Prove that at least 3 of those 9 lines pass through the same point

4 Let a and b be distinct positive integers, bigger that 10^6, such that $(a+b)^3$ is divisible with ab. Prove that $\mid a-b \mid > 10^4$

– Grade 10

1 If $\mid ax^2+bx+c \mid \leq 1$ for all $x \in [-1,1]$ prove that: $a)$ $\mid c \mid \leq 1$ $b)$ $\mid a+c \mid \leq 1$ $c)$ $a^2+b^2+c^2 \leq 5$

2 Let a and b be two positive integers such that $2ab$ divides a^2+b^2-a. Prove that a is perfect square

3 Let AB be a diameter of semicircle h. On this semicircle there is point C, distinct from points A and B. Foot of perpendicular from point C to side AB is point D. Circle k is outside the triangle ADC and at the same time touches semicircle h and sides AB and CD. Touching point of k with side AB is point E, with semicircle h is point T and with side CD is point S $a)$ Prove that points A, S and T are collinear $b)$ Prove that $AC=AE$

4 Let A be a set of 65 integers with pairwise different remainders modulo 2016. Prove that exists a subset $B=\{a,b,c,d\}$ of set A such that $a+b-c-d$ is divisible with 2016

– Grade 11

1 Let a and b be real numbers bigger than 1. Find maximal value of $c \in \mathbb{R}$ such that

$$\frac{1}{3+\log_a b} + \frac{1}{3+\log_b a} \geq c$$

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2 Does there exist a right angled triangle, which hypotenuse is 2016^{2017} and two other sides positive integers.

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3 h_a, h_b and h_c are altitudes, t_a, t_b and t_c are medians of acute triangle, r radius of incircle, and R radius of circumcircle of acute triangle ABC. Prove that

$$\frac{t_a}{h_a} + \frac{t_b}{h_b} + \frac{t_c}{h_c} \leq 1 + \frac{R}{r}$$

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4 It is given circle with center in center of coordinate center with radius of 2016. On circle and inside it are 540 points with integer coordinates such that no three of them are collinear. Prove that there exist two triangles with vertices in given points such that they have same area

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– Grade 12

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1 Let $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{2a_n}$ for $n \geq 1$. Prove that $a)$ $n \leq a_n^2 < n + \sqrt[3]{n}$ $b)$ $\lim_{n\to\infty}(a_n - \sqrt{n}) = 0$

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2 Find all elements $n \in A = \{2, 3, ..., 2016\} \subset \mathbb{N}$ such that:
every number $m \in A$ smaller than n, and coprime with n, must be a prime number

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3 Circle of radius R_1 is inscribed in an acute angle α. Second circle with radius R_2 touches one of the sides forming the angle α in same point as first circle and intersects the second side in points A and B, such that centers of both circles lie inside angle α. Prove that

$$AB = 4\cos\frac{\alpha}{2}\sqrt{(R_2 - R_1)\left(R_1\cos^2\frac{\alpha}{2} + R_2\sin^2\frac{\alpha}{2}\right)}$$

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4 Find all functions $f : \mathbb{Q} \to \mathbb{R}$ such that: $a)$ $f(1) + 2 > 0$ $b)$ $f(x+y) - xf(y) - yf(x) = f(x)f(y) + f(x) + f(y) + xy$, $\forall x, y \in \mathbb{Q}$ $c)$ $f(x) = 3f(x+1) + 2x + 5$, $\forall x \in \mathbb{Q}$
