Regional Olympiad - Federation of Bosnia and Herzegovina 2015

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by gobathegreat, orl

– Sarajevo, April 26th

– Grade 9

1 Find all positive integers a and b such that $ab+1 \mid a^2-1$

2 Let a, b and c be positive real numbers such that $abc=2015$. Prove that

$$\frac{a+b}{a^2+b^2}+\frac{b+c}{b^2+c^2}+\frac{c+a}{c^2+a^2} \leq \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{2015}}$$

3 In parallelogram $ABCD$ holds $AB=BD$. Let K be a point on AB, different from A, such that $KD=AD$. Let M be a point symmetric to C with respect to K, and N be a point symmetric to point B with respect to A. Prove that $DM=DN$

4 Alice and Mary were searching attic and found scale and box with weights. When they sorted weights by mass, they found out there exist 5 different groups of weights. Playing with the scale and weights, they discovered that if they put any two weights on the left side of scale, they can find other two weights and put on to the right side of scale so scale is in balance. Find the minimal number of weights in the box

– Grade 10

1 Solve the inequation:

$$5 \mid x \mid \leq x(3x+2-2\sqrt{8-2x-x^2})$$

2 Let a, b and c be positive real numbers such that $abc=1$. Prove the inequality:

$$\frac{1}{a+b}+\frac{1}{b+c}+\frac{1}{c+a} \leq \frac{a^2+b^2+c^2}{2}$$

3 Let ABC be a triangle with incenter I. Line AI intersects circumcircle of ABC in points A and D, $(A \neq D)$. Incircle of ABC touches side BC in point E . Line DE intersects circumcircle of ABC in points D and F, $(D \neq F)$. Prove that $\angle AFI = 90^\circ$

4 On competition there were 67 students. They were solving 6 problems. Student who solves kth problem gets k points, while student who solves incorrectly kth problem gets $-k$ points. $a)$ Prove that there exist two students with exactly the same answers to problems $b)$ Prove that there exist at least 4 students with same number of points

– Grade 11

1 For real numbers x, y and z, solve the system of equations:

$$x^3 + y^3 = 3y + 3z + 4$$

$$y^3 + z^3 = 3z + 3x + 4$$

$$x^3 + z^3 = 3x + 3y + 4$$

2 Find all triplets (p, a, b) of positive integers such that

$$p = b\sqrt{\frac{a-8b}{a+8b}}$$

is prime

3 Let F be an intersection point of altitude CD and internal angle bisector AE of right angled triangle ABC, $\angle ACB = 90^{\circ}$. Let G be an intersection point of lines ED and BF. Prove that area of quadrilateral $CEFG$ is equal to area of triangle BDG

4 There are 10001 students at an university. Some students join together to form several clubs (a student may belong to different clubs). Some clubs join together to form several societies (a club may belong to different societies). There are a total of k societies. Suppose that the following conditions hold:

i.) Each pair of students are in exactly one club.

ii.) For each student and each society, the student is in exactly one club of the society.

iii.) Each club has an odd number of students. In addition, a club with $2m+1$ students (m is a positive integer) is
in exactly m societies.

Find all possible values of k.

Proposed by Guihua Gong, Puerto Rico

– Grade 12

1 Let a, b, c and d be real numbers such that $a+b+c+d=8$. Prove the inequality:

$$\frac{a}{\sqrt[3]{8+b-d}}+\frac{b}{\sqrt[3]{8+c-a}}+\frac{c}{\sqrt[3]{8+d-b}}+\frac{d}{\sqrt[3]{8+a-c}} \geq 4$$

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2 For positive integer n, find all pairs of coprime integers p and q such that $p+q^2=(n^2+1)p^2+q$

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3 Let O and I be circumcenter and incenter of triangle ABC. Let incircle of ABC touches sides BC, CA and AB in points D, E and F, respectively. Lines FD and CA intersect in point P, and lines DE and AB intersect in point Q. Furthermore, let M and N be midpoints of PE and QF. Prove that $OI \perp MN$

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4 It is given set $A=\{1,2,3,...,2n-1\}$. From set A, at least $n-1$ numbers are expelled such that: $a)$ if number $a \in A$ is expelled, and if $2a \in A$ then $2a$ must be expelled $b)$ if $a,b \in A$ are expelled, and $a+b \in A$ then $a+b$ must be also expelled
Which numbers must be expelled such that sum of numbers remaining in set stays minimal

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