Regional Olympiad - Federation of Bosnia and Herzegovina 2010

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– Grade 9

1 For real numbers a, b, c and d holds:

$$a+b+c+d=0$$

$$a^3+b^3+c^3+d^3=0$$

Prove that sum of some two numbers a, b, c and d is equal to zero

2 In convex quadrilateral $ABCD$, diagonals AC and BD intersect at point O at angle 90°. Let K, L, M and N be orthogonal projections of point O to sides AB, BC, CD and DA of quadrilateral $ABCD$. Prove that $KLMN$ is cyclic

3 If a and b are positive integers such that $ab \mid a^2+b^2$ prove that $a=b$

4 In table of dimensions $2n \times 2n$ there are positive integers not greater than 10, such that numbers lying in unit squares with common vertex are coprime. Prove that there exist at least one number which occurs in table at least $\frac{2n^2}{3}$ times

– Grade 10

1 Find all real numbers (x,y) satisfying the following:

$$x+\frac{3x-y}{x^2+y^2}=3$$

$$y-\frac{x+3y}{x^2+y^2}=0$$

2 It is given acute triangle ABC with orthocenter at point H. Prove that

$$AH \cdot h_a + BH \cdot h_b + CH \cdot h_c = \frac{a^2+b^2+c^2}{2}$$

where a, b and c are sides of a triangle, and h_a, h_b and h_c altitudes of ABC

3 Problem 3 for grade 9

4 It is given set with n^2 elements $(n \geq 2)$ and family $\mathbb{F}$ of subsets of set A, such that every one of them has n elements. Assume that every two sets from $\mathbb{F}$ have at most one common element. Prove that $i)$ Family $\mathbb{F}$ has at most n^2+n elements $ii)$ Upper bound can be reached for $n=3$

– Grade 11

1 Prove the inequality

$$\frac{y^2-x^2}{2x^2+1}+\frac{z^2-y^2}{2y^2+1}+\frac{x^2-z^2}{2z^2+1} \geq 0$$

where x, y and z are real numbers

2 Angle bisector from vertex A of acute triangle ABC intersects side BC in point D, and circumcircle of ABC in point E (different from A). Let F and G be foots of perpendiculars from point D to sides AB and AC. Prove that area of quadrilateral $AEFG$ is equal to the area of triangle ABC

3 Let n be an odd positive integer bigger than 1. Prove that 3^n+1 is not divisible with n

4 In plane there are n noncollinear points A_1, A_2,...,A_n. Prove that there exist a line which passes through exactly two of these points

– Grade 12

1 It is given positive real number a such that:

$$\left\{\frac{1}{a}\right\} = \{a^2\}$$

$$2 < a^2 < 3$$

Find the value of

$$a^{12} - \frac{144}{a}$$

2 Problem 3 for grade 11

3 Problem 4 for grade 11

4 Let AA_1, BB_1 and CC_1 be altitudes of triangle ABC and let A_1A_2, B_1B_2 and C_1C_2 be diameters of Euler circle of triangle ABC. Prove that lines AA_2, BB_2 and CC_2 are concurrent