Romania Team Selection Test 1978

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– Day 1

1 Prove that for every partition of $\{1,2,3,4,5,6,7,8,9\}$ into two subsets, one of the subsets contains three numbers such that the sum of two of them is equal to the double of the third.

2 Suppose that k,l are natural numbers such that $\gcd(11m-1,k)=\gcd(11m-1,l)$, for any natural number m.
Prove that there exists an integer n such that $k=11^n l$.

3 Let $P[X,Y]$ be a polynomial of degree at most 2. If A,B,C,A',B',C' are distinct roots of P such that A,B,C are not collinear and A',B',C' lie on the lines BC,CA, respectively, AB, in the planar representation of these points, show that $P=0$.

4 Diagonals AC and BD of a convex quadrilateral $ABCD$ intersect a point O. Prove that if triangles OAB,OBC,OCD and ODA have the same perimeter, then $ABCD$ is a rhombus. What happens if O is some other point inside the quadrilateral?

5 Prove that there is no square with its four vertices on four concentric circles whose radii form an arithmetic progression.

6 Show that there is no polyhedron whose projection on the plane is a nondegenerate triangle.

7 Let P,Q,R be polynomials of degree 3 with real coefficients such that $P(x)\le Q(x)\le R(x)$, for every real x. Suppose $P-R$ admits a root. Show that $Q=kP+(1-k)R$, for some real number $k\in[0,1]$. What happens if P,Q,R are of degree 4, under the same circumstances?

8 For any set A we say that two functions $f,g:A\longrightarrow A$ are *similar,* if there exists a bijection $h:A\longrightarrow A$ such that $f\circ h=h\circ g$.

a) If A has three elements, construct a finite, arbitrary number functions, having as domain and codomain A, that are two by two similar, and every other function with the same domain and codomain as the ones determined is similar to, at least, one of them.
b) For $A=\mathbb{R}$, show that the functions $\sin$ and $-\sin$ are similar.

9 A sequence $(x_n)_{n\ge 0}$ of real numbers satisfies $x_0>1=x_{n+1}\left(x_n-\lfloor x_n\rfloor\right)$, for each $n\ge 1$.
Prove that if $(x_n)_{n\ge 0}$ is periodic, then x_0 is a root of a quadratic equation. Study the converse.

– Day 2

1 Associate to any point (h,k) in the integer net of the cartesian plane a real number $a_{h,k}$ so that

$$a_{h,k} = \frac{1}{4}\left(a_{h-1,k} + a_{h+1,k} + a_{h,k-1} + a_{h,k+1}\right), \quad \forall h,k \in \mathbb{Z}.$$

a) Prove that its possible that all the elements of the set $A := \left\{ a_{h,k} \middle| h,k \in \mathbb{Z} \right\}$ are different.
b) If so, show that the set A hasnt any kind of boundary.

2 Prove that there is a function $F : \mathbb{N} \longrightarrow \mathbb{N}$ satisfying $(F \circ F)(n) = n^2$, for all $n \in \mathbb{N}$.

3 Let $A_1, A_2, ..., A_{3n}$ be $3n \geq 3$ planar points such that $A_1A_2A_3$ is an equilateral triangle and $A_{3k+1}, A_{3k+2}, A_{3k+3}$ are the midpoints of the sides of $A_{3k-2}A_{3k-1}A_{3k}$, for all $1 \leq k < n$. Of two different colors, each one of these points are colored, either with one, either with another.

a) Prove that, if $n \geq 7$, then some of these points form a monochromatic (only one color) isosceles trapezoid.
b) What about $n = 6$?

4 Let $\mathcal{M}$ a set of $3n \geq 3$ planar points such that the maximum distance between two of these points is 1. Prove that:

a) among any four points,there are two aparted by a distance at most $\frac{1}{\sqrt{2}}$.
b) for $n = 2$ and any $\epsilon > 0$, it is possible that 12 or 15 of the distances between points from $\mathcal{M}$ lie in the interval $(1-\epsilon, 1]$; but any 13 of the distances cant be found all in the interval $\left(\frac{1}{\sqrt{2}}, 1\right]$.
c) there exists a circle of diameter $\sqrt{6}$ that contains $\mathcal{M}$.
d) some two points of $\mathcal{M}$ are on a distance not exceeding $\frac{4}{3\sqrt{n} - \sqrt{3}}$.

– Day 3

1 In a convex quadrilateral $ABCD$, let A, B be the orthogonal projections to CD of A, respectively, B.

a) Assuming that $BB \leq AA$ and that the perimeter of $ABCD$ is $(AB + CD) \cdot BB$, is $ABCD$ necessarily a trapezoid?
b) The same question with the addition that $\angle BAD$ is obtuse.

2 Points A, B, C are arbitrarily taken on edges SA, SB, respectively, SC of a tetrahedron $SABC$. Plane forrmed by ABC intersects the plane ρ, formed by ABC, in a line d. Prove that, meanwhile the plane ρ rotates around d, the lines AA, BB and CC are, and remain concurrent. Find de locus of the respective intersections.

3 **a)** Let D_1, D_2, D_3 be pairwise skew lines. Through every point $P_2 \in D_2$ there is an unique common secant of these three lines that intersect D_1 at P_1 and D_3 at P_3. Let coordinate systems be introduced on D_2 and D_3 having as origin O_2, respectively, O_3. Find a relation between the coordinates of P_2 and P_3.

b) Show that there exist four pairwise skew lines with exactly two common secants. Also find examples with exactly one and with no common secants.

c) Let F_1, F_2, F_3, F_4 be any four secants of D_1, D_2, D_3. Prove that F_1, F_2, F_3, F_4 have infinitely many common secants.

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4 Solve the equation $\sin x \sin 2x \cdots \sin nx + \cos x \cos 2x \cdots \cos nx = 1$, for $n \in \mathbb{N}$ and $x \in \mathbb{R}$.

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5 Find locus of points M inside an equilateral triangle ABC such that

$$\angle MBC + \angle MCA + \angle MAB = \pi/2.$$

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6 **a)** Prove that $0 = \inf\{|x\sqrt{2} + y\sqrt{3} + y\sqrt{5}| | x, y, z \in \mathbb{Z}, x^2 + y^2 + z^2 > 0\}$

b) Prove that there exist three positive rational numbers a, b, c such that the expression $E(x, y, z) := xa + yb + zc$ vanishes for infinitely many integer triples (x, y, z), but it doesnt get arbitrarily close to 0.

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7 **a)** Prove that for any natural number $n \geq 1$, there is a set $\mathcal{M}$ of n points from the Cartesian plane such that the barycenter of every subset of $\mathcal{M}$ has integral coordinates (both coordinates are integer numbers).

b) Show that if a set $\mathcal{N}$ formed by an infinite number of points from the Cartesian plane is given such that no three of them are collinear, then there exists a finite subset of $\mathcal{N}$, the barycenter of which has non-integral coordinates.

– Day 4

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1 Show that for every natural number $a \geq 3$, there are infinitely many natural numbers n such that $a^n \equiv 1 \pmod{n}$. Does this hold for $n = 2$?

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2 Let k be a natural number. A function $f : S := \{x_1, x_2, ..., x_k\} \longrightarrow \mathbb{R}$ is said to be *additive* if, whenever $n_1 x_1 + n_2 x_2 + \cdots + n_k x_k = 0$, it holds that $n_1 f(x_1) + n_2 f(x_2) + \cdots + n_k f(x_k) = 0$, for all natural numbers $n_1, n_2, ..., n_k$.

Show that for every additive function and for every finite set of real numbers T, there exists a second function, which is a real additive function defined on $S \cup T$ and which is equal to the former on the restriction S.

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3 Let p be a natural number and let two partitions $\mathcal{A}=\{A_1, A_2, ..., A_p\}, \mathcal{B}=\{B_1, B_2, ... B_p\}$ of a finite set $\mathcal{M}$. Knowing that, whenever an element of $\mathcal{A}$ doesnt have any elements in common with another of $\mathcal{B}$, it holds that the number of elements of these two is greater than p, prove that $|\mathcal{M}| \geq \frac{1}{2}\left(1+p^2\right)$. Can equality hold?

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4 Let be some points on a plane, no three collinear. We associate a positive or a negative value to every segment formed by these. Prove that the number of points, the number of segments with negative associated value, and the number of triangles that has a negative product of the values of its sides, share the same parity.
