Romania National Olympiad 2000

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– Grade level 9

1 Let be two natural primes $1 \leq q \leq p$. Prove that $\left(\sqrt{p^2+q}+p\right)^2$ is irrational and its fractional part surpasses $3/4$.

2 Let A, B be two points in a plane and let two numbers $a, b \in (0,1)$. For each point M that is not on the line AB consider P on the segment AM and N on BM (both excluding the extremities) such that $BN = b \cdot BM$ and $AP = a \cdot AM$. Find the locus of the points M for which $AN = BP$.

3 Let be a natural number $n \geq 2$ and an expression of n variables

$$E\left(x_1, x_2, ..., x_n\right) = x_1^2 + x_2^2 + \cdots + x_n^2 - x_1x_2 - x_2x_3 - \cdots - x_{n-1}x_n - x_nx_1.$$

Determine $\sup_{x_1,\ldots,x_n \in [0,1]} E\left(x_1, x_2, ..., x_n\right)$ and the specific values at which this supremum is attained.

4 Let I be the center of the incircle of a triangle ABC. Shw that, if for any point M on the segment AB (extremities excluded) there exist two points N, P on BC, respectively, AC (both excluding the extremities) such that the center of mass of MNP coincides with I, then ABC is equilateral.

– Grade level 10

1 Let $(x_n)_{n\geq 1}$ be a sequence having $x_1 = 3$ and defined as $x_{n+1} = \left\lfloor \sqrt{2}x_n \right\rfloor$, for every natural number n. Find all values m for which the terms x_m, x_{m+1}, x_{m+2} are in arithmetic progression, where $\lfloor \rfloor$ denotes the integer part.

2 Demonstrate that if $z_1, z_2 \in \mathbb{C}^*$ satisfy the relation:

$$z_1 \cdot 2^{\left|z_1\right|} + z_2 \cdot 2^{\left|z_2\right|} = (z_1+z_2) \cdot 2^{\left|z_1+z_2\right|},$$

then $z_1^6 = z_2^6$

3 Let be a tetahedron $ABCD$, and E be the projection of D on the plane formed by ABC. If $\mathcal{A}_{\mathcal{R}}$ denotes the area of the region $\mathcal{R}$, show that the following affirmations are equivalent:

a) $C = E \vee CE \parallel AB$

b) $M \in \overline{CD} \implies \mathcal{A}_{ABM}^2 = \frac{CM^2}{CD^2} \cdot \mathcal{A}_{ABD}^2 + \left(1 - \frac{CM^2}{CD^2}\right) \cdot \mathcal{A}_{ABC}^2$

4 Let f be a polynom of degree 3 and having rational coefficients. Prove that, if there exist two distinct nonzero rational numbers a, b and two roots x, y of f such that $ax + by$ is rational, then all roots of f are rational.

– Grade level 11

1 Let $\mathcal{M} = \left\{ A \in M_2(\mathbb{C}) \,\middle|\, \det(A - zI_2) = 0 \implies |z| < 1 \right\}$. Prove that:

$$X, Y \in \mathcal{M} \wedge X \cdot Y = Y \cdot X \implies X \cdot Y \in \mathcal{M}.$$

2 Study the convergence of a sequence $(x_n)_{n\ge 0}$ for which $x_0 \in \mathbb{R}\setminus\mathbb{Q}$, and $x_{n+1} \in \left\{ \frac{x_n+1}{x_n}, \frac{x_n+2}{2x_n-1} \right\}$, for all $n \ge 1$.

3 A function $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is *olympic* if, any finite number of pairwise distinct elements of $\mathbb{R}^2$ at which the function takes the same value represent in the plane the vertices of a convex polygon.

Prove that if p if a complex polynom of degree at least 1, then the function $\mathbb{R}^2 \ni (x, y) \mapsto |p(x + iy)|$ is olympic if and only if the roots of p are all equal.

4 Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function that satisfies the conditions: (i) $\lim_{x\to\infty}(f \circ f)(x) = \infty = -\lim_{x\to-\infty}(f \circ f)(x)$ (ii) f has Darbouxs property

a) Prove that the limits of f at $\pm\infty$ exist.
b) Is possible for the limits from **a)** to be finite?

– Grade level 12

1 Let $a \in (1, \infty)$ and a countinuous function $f : [0, \infty) \longrightarrow \mathbb{R}$ having the property:

$$\lim_{x\to\infty} x f(x) \in \mathbb{R}.$$

a) Show that the integral $\int_1^\infty \frac{f(x)}{x} dx$ and the limit $\lim_{t\to\infty} t \int_1^a f\left(x^t\right) dx$ both exist, are finite and equal.

b) Calculate $\lim_{t\to\infty} t \int_1^a \frac{dx}{1+x^t}$.

2 For any partition P of $[0, 1]$, consider the set

$$\mathcal{A}(P) = \left\{ f : [0,1] \longrightarrow \mathbb{R} \,\middle|\, \exists f\Big|_{[0,1]} \wedge \int_0^1 |f(x)| dx = 1 \wedge (y \in P \implies f(y) = 0) \right\}.$$

Prove that there exists a partition P_0 of $[0,1]$ such that

$$g \in \mathcal{A}(P_0) \implies \sup_{x\in[0,1]} \left|g(x)\right| > 4 \cdot \#P.$$

Here, $\#D$ denotes the natural number d such that $0 = x_0 < x_1 < \cdots < x_d = 1$ is a partition D of $[0,1]$.

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3 We say that the abelian group G has property *(P)* if, for any commutative group H, any $H \leq H$ and any momorphism $\mu : H \longrightarrow G$, there exists a morphism $\mu : H \longrightarrow G$ such that $\mu\Big|_H = \mu$. Show that:

a) the group $(\mathbb{Q}^*, \cdot)$ hasnt property *(P)*.
b) the group $(\mathbb{Q}, +)$ has property *(P)*.

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4 Prove that a nontrivial finite ring is not a skew field if and only if the equation $x^n + y^n = z^n$ has nontrivial solutions in this ring for any natural number n.
