USA Team Selection Test for EGMO 2019

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TST#1 Thursday, December 6th, 2018

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1 A 3×3 grid of unit cells is given. A [i]snake of length k[/i] is an animal which occupies an ordered k-tuple of cells in this grid, say $(s_1, \dots, s_k)$. These cells must be pairwise distinct, and s_i and s_{i+1} must share a side for $i = 1, \dots, k-1$. After being placed in a finite $n \times n$ grid, if the snake is currently occupying $(s_1, \dots, s_k)$ and s is an unoccupied cell sharing a side with s_1, the snake can *move* to occupy $(s, s_1, \dots, s_{k-1})$ instead. The snake has *turned around* if it occupied $(s_1, s_2, \dots, s_k)$ at the beginning, but after a finite number of moves occupies $(s_k, s_{k-1}, \dots, s_1)$ instead.

Find the largest integer k such that one can place some snake of length k in a 3×3 grid which can turn around.

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2 Let ABC be a triangle and let M and N denote the midpoints of $\overline{AB}$ and $\overline{AC}$, respectively. Let X be a point such that $\overline{AX}$ is tangent to the circumcircle of triangle ABC. Denote by ω_B the circle through M and B tangent to $\overline{MX}$, and by ω_C the circle through N and C tangent to $\overline{NX}$. Show that ω_B and ω_C intersect on line BC.

Merlijn Staps

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3 Let n be a positive integer such that the number

$$\frac{1^k + 2^k + \dots + n^k}{n}$$

is an integer for any $k \in \{1, 2, \dots, 99\}$. Prove that n has no divisors between 2 and 100, inclusive.

TST#2 Thursday, January 17th, 2019

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4 For every pair (m, n) of positive integers, a positive real number $a_{m,n}$ is given. Assume that

$$a_{m+1,n+1} = \frac{a_{m,n+1}a_{m+1,n} + 1}{a_{m,n}}$$

for all positive integers m and n. Suppose further that $a_{m,n}$ is an integer whenever $\min(m, n) \le 2$. Prove that $a_{m,n}$ is an integer for all positive integers m and n.

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5 Let the excircle of a triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define points B_1 on $\overline{CA}$ and C_1 on $\overline{AB}$ analogously, using the excircles opposite B and

C, respectively. Denote by γ the circumcircle of triangle $A_1B_1C_1$ and assume that γ passes through vertex A.

- Show that $\overline{AA_1}$ is a diameter of γ.
- Show that the incenter of $\triangle ABC$ lies on line B_1C_1.

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6 Let n be a positive integer. Tasty and Stacy are given a circular necklace with $3n$ sapphire beads and $3n$ turquoise beads, such that no three consecutive beads have the same color. They play a cooperative game where they alternate turns removing three consecutive beads, subject to the following conditions:

-Tasty must remove three consecutive beads which are turquoise, sapphire, and turquoise, in that order, on each of his turns.
-Stacy must remove three consecutive beads which are sapphire, turquoise, and sapphire, in that order, on each of her turns.

They win if all the beads are removed in $2n$ turns. Prove that if they can win with Tasty going first, they can also win with Stacy going first.

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