AoPS Online
Art of Problem Solving

Brazil National Olympiad 2018

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Day 1 Tuesday, November 13

1 We say that a polygon P is *inscribed* in another polygon Q when all vertices of P belong to perimeter of Q. We also say in this case that Q is *circumscribed* to P. Given a triangle T, let l be the maximum value of the side of a square inscribed in T and L be the minimum value of the side of a square circumscribed to T. Prove that for every triangle T the inequality $L/l \geq 2$ holds and find all the triangles T for which the equality occurs.

2 Azambuja writes a rational number q on a blackboard. One operation is to delete q and replace it by $q+1$; or by $q-1$; or by $\frac{q-1}{2q-1}$ if $q \neq \frac{1}{2}$. The final goal of Azambuja is to write the number $\frac{1}{2018}$ after performing a finite number of operations.

a) Show that if the initial number written is 0, then Azambuja cannot reach his goal.

b) Find all initial numbers for which Azambuja can achieve his goal.

3 Let k, n be fixed positive integers. In a circular table, there are placed pins numbered successively with the numbers $1, 2 \ldots, n$, with 1 and n neighbors. It is known that pin 1 is golden and the others are white. Arnaldo and Bernaldo play a game, in which a ring is placed initially on one of the pins and at each step it changes position. The game begins with Bernaldo choosing a starting pin for the ring, and the first step consists of the following: Arnaldo chooses a positive integer d any and Bernaldo moves the ring d pins clockwise or counterclockwise (positions are considered modulo n, i.e., pins x, y equal if and only if n divides $x-y$). After that, the ring changes its position according to one of the following rules, to be chosen at every step by Arnaldo:

Rule 1: Arnaldo chooses a positive integer d and Bernaldo moves the ring d pins clockwise or counterclockwise.

Rule 2: Arnaldo chooses a direction (clockwise or counterclockwise), and Bernaldo moves the ring in the chosen direction in d or kd pins, where d is the size of the last displacement performed.

Arnaldo wins if, after a finite number of steps, the ring is moved to the golden pin. Determine, as a function of k, the values of n for which Arnaldo has a strategy that guarantees his victory, no matter how Bernaldo plays.

Day 2 Wednesday, November 14

4 Esmeralda writes $2n$ real numbers $x_1, x_2, \ldots, x_{2n}$, all belonging to the interval $[0, 1]$, around a circle and multiplies all the pairs of numbers neighboring to each other, obtaining, in the counterclockwise direction, the products $p_1 = x_1x_2$, $p_2 = x_2x_3$, $\ldots$, $p_{2n} = x_{2n}x_1$. She adds the products with even indices and subtracts the products with odd indices. What is the maximum possible number Esmeralda can get?

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5 Consider the sequence in which $a_1 = 1$ and a_n is obtained by juxtaposing the decimal representation of n at the end of the decimal representation of a_{n-1}. That is, $a_1 = 1$, $a_2 = 12$, $a_3 = 123$, $\dots$, $a_9 = 123456789$, $a_{10} = 12345678910$ and so on. Prove that infinitely many numbers of this sequence are multiples of 7.

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6 Consider $4n$ points in the plane, with no three points collinear. Using these points as vertices, we form $\binom{4n}{3}$ triangles. Show that there exists a point X of the plane that belongs to the interior of at least $2n^3$ of these triangles.

Level 2 -

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1 Every day from day 2, neighboring cubes (cubes with common faces) to red cubes also turn red and are numbered with the day number.

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2 We say that a quadruple (A, B, C, D) is *dobarulho* when A, B, C are non-zero algorisms and D is a positive integer such that: $1.\ A \leq 8$ $2.\ D > 1$ $3.\ D$ divides the six numbers $\overline{ABC}$, $\overline{BCA}$, $\overline{CAB}$, $\overline{(A+1)CB}$, $\overline{CB(A+1)}$, $\overline{B(A+1)C}$.
Find all such quadruples.

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3 Let ABC be an acute-angled triangle with circumcenter O and orthocenter H. The circle with center X_a passes in the points A and H and is tangent to the circumcircle of ABC. Define X_b, X_c analogously, let O_a, O_b, O_c the symmetric of O to the sides BC, AC and AB, respectively. Prove that the lines O_aX_a, O_bX_b, O_cX_c are concurrents.

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4 a) In a XYZ triangle, the incircle tangents the XY and XZ sides at the T and W points, respectively. Prove that:

$$XT = XW = \frac{XY + XZ - YZ}{2}$$

Let ABC be a triangle and D is the foot of the relative height next to A. Are I and J the incentives from triangle ABD and ACD, respectively. The circles of ABD and ACD tangency AD at points M and N, respectively. Let P be the tangency point of the BC circle with the AB side. The center circle A and radius AP intersect the height D at K.
b) Show that the triangles IMK and KNJ are congruent
c) Show that the $IDJK$ quad is inscribited

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5 One writes, initially, the numbers $1, 2, 3, \dots, 10$ in a board. An operation is to delete the numbers a, b and write the number $a + b + \frac{ab}{f(a,b)}$, where $f(a, b)$ is the sum of all numbers in the board excluding a and b, one will make this until remain two numbers x, y with $x \geq y$. Find the maximum value of x.

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6 Let $S(n)$ be the sum of digits of n. Determine all the pairs (a, b) of positive integers, such that the expression $S(an + b) - S(n)$ has a finite number of values, where n is varying in the positive integers.