Korea National Olympiad 2018

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– day 1

1 Let there be an acute triangle $\triangle ABC$ with incenter I. E is the foot of the perpendicular from I to AC. The line which passes through A and is perpendicular to BI hits line CI at K. The line which passes through A and is perpendicular to CI hits the line which passes through C and is perpendicular to BI at L. Prove that E, K, L are colinear.

2 For a positive integer n, denote $p(n)$ to be the number of nonnegative integer tuples (x,y,z,w) such that $x+y+2z+3w=n-1$. Also, denote $q(n)$ to be the number of nonnegative integer tuples (a,b,c,d) such that

(i). $a+b+c+d=n$.
(ii). $a \geq b$, $c \geq d$, $a \geq d$.
(iii). $b < c$.

Prove that for all n, $p(n)=q(n)$.

3 Denote $f(x)=x^4+2x^3-2x^2-4x+4$. Prove that there are infinitely many primes p that satisfies the following.

For all positive integers m, $f(m)$ is not a multiple of p.

4 Find all real values of K which satisfies the following.

Let there be a sequence of real numbers $\{a_n\}$ which satisfies the following for all positive integers n.

(i). $0 < a_n < n^K$.
(ii). $a_1+a_2+\cdots+a_n < \sqrt{n}$.

Then, there exists a positive integer N such that for all integers $n > N$,

$$a_1^{2018}+a_2^{2018}+\cdots+a_n^{2018} < \frac{n}{2018}$$

– day 2

5 Let there be a convex quadrilateral $ABCD$. The angle bisector of $\angle A$ meets the angle bisector of $\angle B$, the angle bisector of $\angle D$ at P, Q respectively. The angle bisector of $\angle C$ meets the angle bisector of $\angle D$, the angle bisector of $\angle B$ at R, S respectively. P, Q, R, S are all distinct points. PR and QS meets perpendicularly at point Z. Denote l_A, l_B, l_C, l_D as the exterior angle bisectors of $\angle A, \angle B, \angle C, \angle D$. Denote $E = l_A \cap l_B$, $F = l_B \cap l_C$, $G = l_C \cap l_D$, and $H = l_D \cap l_A$. Let K, L, M, N be the midpoints of FG, GH, HE, EF respectively.

Prove that the area of quadrilateral $KLMN$ is equal to $ZM \cdot ZK + ZL \cdot ZN$.

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6 Let $n \geq 3$ be a positive integer. For every set S with n distinct positive integers, prove that there exists a bijection $f : \{1, 2, \cdots n\} \to S$ which satisfies the following condition.

For all $1 \leq i < j < k \leq n$, $f(j)^2 \neq f(i) \cdot f(k)$.

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7 Let there be a figure with 9 disks and 11 edges, as shown below.

We will write a real number in each and every disk. Then, for each edge, we will write the square of the difference between the two real numbers written in the two disks that the edge connects. We must write 0 in disk A, and 1 in disk I. Find the minimum sum of all real numbers written in 11 edges.

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8 Let there be positive integers a, c. Positive integer b is a divisor of $ac - 1$. For a positive rational number r which is less than 1, define the set $A(r)$ as follows.

$$A(r) = \{m(r - ac) + nab | m, n \in \mathbb{Z}\}$$

Find all rational numbers r which makes the minimum positive rational number in $A(r)$ greater than or equal to $\frac{ab}{a+b}$.
