AoPS Online
Art of Problem Solving

The long-lost problems of the 2006 iTest are finally here!
www.artofproblemsolving.com/community/c779283
by RockmanEX3

– **Multiple Choice Problems**

– Find the number of positive integral divisors of 2006.

(A) 8

– Find the harmonic mean of 10 and 20.

(A) 15 **(B)** $\frac{40}{3}$

– Let I, T, E, S be distinct positive integers such that the product $ITEST = 2006$. What is the largest possible value of the sum $I+T+E+S+T+2006$?

(A) 2086 **(B)** 4012 **(C)** 2144

– Four couples go ballroom dancing one evening. Their first names are Henry, Peter, Louis, Roger, Elizabeth, Jeanne, Mary, and Anne. If Henry's wife is not dancing with her husband (but with Elizabeth's husband), Roger and Anne are not dancing, Peter is playing the trumpet, and Mary is playing the piano, and Anne's husband is not Peter, who is Roger's wife?

(A) Elizabeth **(B)** Jeanne **(C)** Mary **(D)** Anne

– A line has y-intercept $(0, 3)$ and forms a right angle to the line $2x+y=3$. Find the x-intercept of the line.

(A) $(4, 0)$ **(B)** $(6, 0)$ **(C)** $(-4, 0)$ **(D)** $(-6, 0)$ **(E)** none of the above

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– What is the remainder when 2^{2006} is divided by 7?

(A) 0 **(B)** 1 **(C)** 2 **(D)** 3 **(E)** 4 **(F)** 5

– The sum of 17 consecutive integers is 2006. Find the second largest integer.

(A) 17 **(B)** 72 **(C)** 95 **(D)** 101 **(E)** 102 **(F)** 111 **(G)** 125

– The point P is a point on a circle with center O. Perpendicular lines are drawn from P to perpendicular diameters, AB and CD, meeting them at points Y and Z, respectively. If the diameter of the circle is 16, what is the length of YZ?

$\text{(A)}\,4$ $\text{(B)}\,8$ $\text{(C)}\,6\sqrt{3}$ $\text{(D)}\,4\sqrt{3}$ $\text{(E)}\,4\sqrt{2}$ $\text{(F)}\,12$ $\text{(G)}\,6$ (H) none of the above

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– If $\sin(x) = -\frac{5}{13}$ and x is in the third quadrant, what is the absolute value of $\cos(\frac{x}{2})$?

$\text{(A)}\ \frac{\sqrt{3}}{3}\qquad\text{(B)}\ \frac{2\sqrt{3}}{3}\qquad\text{(C)}\ \frac{6}{13}\qquad\text{(D)}\ \frac{5}{13}\qquad\text{(E)}\ -\frac{5}{13}$
$\text{(F)}\ \frac{\sqrt{26}}{26}\qquad\text{(G)}\ -\frac{\sqrt{26}}{26}\qquad\text{(H)}\ \frac{\sqrt{2}}{2}\qquad\text{(I)}$ none of the above

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– Find the number of elements in the first 64 rows of Pascal's Triangle that are divisible by 4.

$\text{(A)}\ 256\qquad\text{(B)}\ 496\qquad\text{(C)}\ 512\qquad\text{(D)}\ 640\qquad\text{(E)}\ 796$
$\text{(F)}\ 946\qquad\text{(G)}\ 1024\qquad\text{(H)}\ 1134\qquad\text{(I)}\ 1256\qquad\text{(J)}$ none of the above

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– Find the radius of the inscribed circle of a triangle with sides of length 13, 30, and 37.

(A) $\frac{9}{2}$ (B) $\frac{7}{2}$ (C) 4 (D) $-\sqrt{2}$ (E) $4\sqrt{5}$ (F) 6
(G) $\frac{11}{2}$ (H) $\frac{13}{2}$ (I) none of the above (J) 1 (K) no triangle exists

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– What is the highest possible probability of getting 12 of these 20 multiple choice questions correct, given that you don't know how to work any of them and are forced to blindly guess on each one?

Note: The first question has 1 choice, the second question has 2 choices, and so on.

(A) $\frac{1}{6!}$ (B) $\frac{1}{7!}$ (C) $\frac{1}{8!}$ (D) $\frac{1}{9!}$ (E) $\frac{1}{10!}$ (F) $\frac{1}{11!}$

(G) $\frac{1}{12!}$ (H) $\frac{2}{8!}$ (I) $\frac{2}{10!}$ (J) $\frac{2}{12!}$ (K) $\frac{1}{20!}$ (L) none of the above

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– Suppose that x, y, z are three distinct prime numbers such that $x + y + z = 49$. Find the maximum possible value for the product xyz.

(A) 615 (B) 1295 (C) 2387 (D) 1772 (E) 715 (F) 442 (G) 1479
(H) 2639 (I) 3059 (J) 3821 (K) 3145 (L) 1715 (M) none of the above

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– Find x, where x is the smallest positive integer such that 2^x leaves a remainder of 1 when divided by 5, 7, and 31.

(A) 15 (B) 20 (C) 25 (D) 30 (E) 28 (F) 32 (G) 64
(H) 128 (I) 45 (J) 50 (K) 60 (L) 70 (M) 80 (N) none of the above

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– How many integers between 1 and 2006, inclusive, are perfect squares?

(A) 37 (B) 38 (C) 39 (D) 40 (E) 41 (F) 42 (G) 43 (H) 44

(I) 45 (J) 46 (K) 47 (L) 48 (M) 49 (N) 50 (O) none of the above

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– The Minnesota Twins face the New York Mets in the 2006 World Series. Assuming the two teams are evenly matched (each has a $.5$ probability of winning any game) what is the probability that the World Series (a best of 7 series of games which lasts until one team wins four games) will require the full seven games to determine a winner?

(A) $\frac{1}{16}$ (B) $\frac{1}{8}$ (C) $\frac{3}{16}$ (D) $\frac{1}{4}$ (E) $\frac{5}{16}$

(F) $\frac{3}{8}$ (G) $\frac{5}{32}$ (H) $\frac{7}{32}$ (I) $\frac{9}{32}$ (J) $\frac{3}{64}$ (K) $\frac{5}{64}$

(L) $\frac{7}{64}$ (M) $\frac{1}{2}$ (N) $\frac{13}{32}$ (O) $\frac{11}{32}$ (P) none of the above

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– Let $\sin(2x) = \frac{1}{7}$. Find the numerical value of $\sin(x)\sin(x)\sin(x)\sin(x)+\cos(x)\cos(x)\cos(x)\cos(x)$.

(A) $\frac{2305}{2401}$ (B) $\frac{4610}{2401}$ (C) $\frac{2400}{2401}$ (D) $\frac{6915}{2401}$ (E) $\frac{1}{2401}$ (F) 0

(G) $\frac{195}{196}$ (H) $\frac{195}{98}$ (I) $\frac{97}{98}$ (J) $\frac{1}{49}$ (K) $\frac{2}{49}$ (L) $\frac{48}{49}$

(M) $\frac{96}{49}$ (N) π (O) none of the above (P) 1 (Q) 2

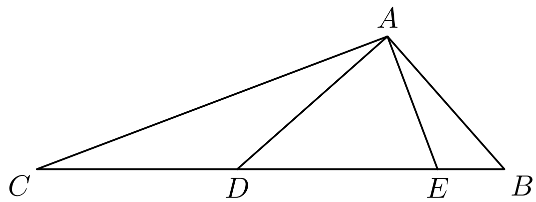
– **Free Response Problems**

– What is the last (rightmost) digit of 3^{2006}?

– Triangle ABC has sidelengths $AB = 75$, $BC = 100$, and $CA = 125$. Point D is the foot of the altitude from B, and E lies on segment BC such that $DE \perp BC$. Find the area of the triangle BDE.

– Jack and Jill are playing a chance game. They take turns alternately rolling a fair six sided die labeled with the integers 1 through 6 as usual (fair meaning the numbers appear with equal probability.) Jack wins if a prime number appears when he rolls, while Jill wins if when she rolls a number greater than 1 appears. The game terminates as soon as one of them has won. If Jack rolls first in a game, then the probability of that Jill wins the game can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Compute $m + n$.

– Points D and E are chosen on side BC of triangle ABC such that E is between B and D and $BE = 1$, $ED = DC = 3$. If $\angle BAD = \angle EAC = 90^\circ$, the area of ABC can be expressed as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is a positive integer not divisible by the square of any prime. Compute $p + q + r$.



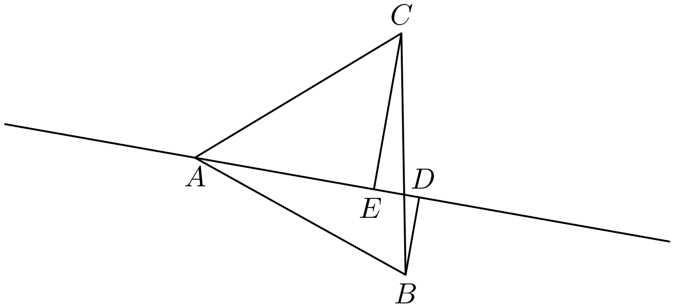
– The expression

$$\frac{(1+2+\cdots+10)(1^3+2^3+\cdots+10^3)}{(1^2+2^2+\cdots+10^2)^2}$$

reduces to $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

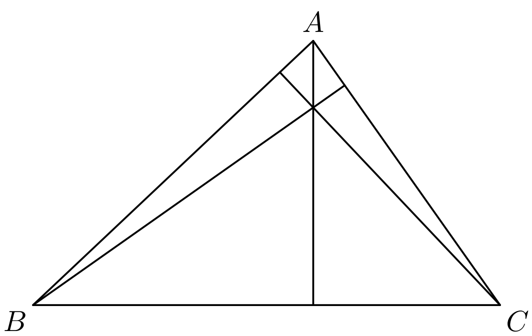
– A rectangle has area A and perimeter P. The largest possible value of $\frac{A}{P^2}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

– Line ℓ passes through A and into the interior of the equilateral triangle ABC. D and E are the orthogonal projections of B and C onto ℓ respectively. If $DE=1$ and $2BD=CE$, then the area of ABC can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Determine $m+n$.

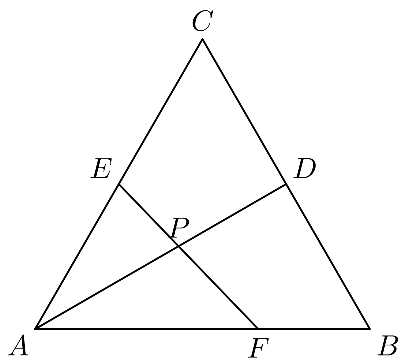


– The largest prime factor of 999999999999 is greater than 2006. Determine the remainder obtained when this prime factor is divided by 2006.

– The altitudes in triangle ABC have lengths 10, 12, and 15. The area of ABC can be expressed as $\frac{m\sqrt{n}}{p}$, where m and p are relatively prime positive integers and n is a positive integer not divisible by the square of any prime. Find $m+n+p$.



– Triangle ABC is equilateral. Points D and E are the midpoints of segments BC and AC respectively. F is the point on segment AB such that $2BF = AF$. Let P denote the intersection of AD and EF, The value of EP/PF can be expressed as m/n where m and n are relatively prime positive integers. Find $m+n$.



– The value of the infinite series

$$\sum_{n=2}^{\infty} \frac{n^4+n^3+n^2-n+1}{n^6-1}$$

can be expressed as $\frac{p}{q}$ where p and q are relatively prime positive numbers. Compute $p+q$.

– Triangle ABC is scalene. Points P and Q are on segment BC with P between B and Q such that $BP = 21$, $PQ = 35$, and $QC = 100$. If AP and AQ trisect $\angle A$, then $\frac{AB}{AC}$ can be written uniquely as $\frac{p\sqrt{q}}{r}$, where p and r are relatively prime positive integers and q is a positive integer not divisible by the square of any prime. Determine $p+q+r$.

– Six students sit in a group and chat during a complicated mathematical lecture. The professor, annoyed by the chatter, splits the group into two or more smaller groups. However, the smaller groups with at least two members continue to produce chatter, so the professor again chooses one noisy group and splits it into smaller groups. This process continues until the professor achieves the silence he needs to teach Algebraic Combinatorics. Suppose the procedure can be carried out in N ways, where the order of group breaking matters (if A and B are disjoint groups, then breaking up group A and then B is considered different form breaking up group B and then A even if the resulting partitions are identical) and where a group of students is treated as an unordered set of people. Compute the remainder obtained when N is divided by 2006.

– For each positive integer n let S_n denote the set of positive integers k such that $n^k - 1$ is divisible by 2006. Define the function $P(n)$ by the rule

$$P(n) := \begin{cases} \min(s)_{s\in S_n} & \text{if } S_n \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let d be the least upper bound of $\{P(1), P(2), P(3), \ldots\}$ and let m be the number of integers i such that $1 \leq i \leq 2006$ and $P(i) = d$. Compute the value of $d + m$.

– Compute the *number* of ordered quadruples (w, x, y, z) of complex numbers (not necessarily nonreal) such that the following system is satisfied:

$$\begin{aligned} wxyz &= 1 \\ wxy^2 + wx^2z + w^2yz + xyz^2 &= 2 \\ wx^2y + w^2y^2 + w^2xz + xy^2z + x^2z^2 + ywz^2 &= -3 \\ w^2xy + x^2yz + wy^2z + wxz^2 &= -1 \end{aligned}$$

– Let α denote $\cos^{-1}(\frac{2}{3})$. The recursive sequence $a_0, a_1, a_2, \ldots$ satisfies $a_0 = 1$ and, for all positive integers n,

$$a_n = \frac{\cos(n\alpha) - (a_1 a_{n-1} + \cdots + a_{n-1} a_1)}{2a_0}.$$

Suppose that the series

$$\sum_{k=0}^{\infty} \frac{a_k}{2^k}$$

can be expressed uniquely as $\frac{p\sqrt{q}}{r}$, where p and r are coprime positive integers and q is not divisible by the square of any prime. Find the value of $p + q + r$.

– The positive reals x, y, z satisfy the relations

$$\begin{aligned} x^2 + xy + y^2 &= 1 \\ y^2 + yz + z^2 &= 2 \\ z^2 + zx + x^2 &= 3. \end{aligned}$$

The value of y^2 can be expressed uniquely as $\frac{m-n\sqrt{p}}{q}$, where m, n, p, q are positive integers such that p is not divisible by the square of any prime and no prime dividing q divides both m and n. Compute $m + n + p + q$.

– Segment AB is a diameter of circle Γ_1. Point C lies in the interior of segment AB such that $BC = 7$, and D is a point on Γ_1 such that $BD = CD = 10$. Segment AC is a diameter of the circle Γ_2. A third circle, ω, is drawn internally tangent to Γ_1, externally tangent to Γ_2, and tangent to segment CD. If ω is centered on the opposite side of CD as B, then the radius of ω can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

– $ABCDEFGHIJKL$ is a regular dodecagon. The number 1 is written at the vertex A, and 0's are written at each of the other vertices. Suddenly and simultaneously, the number at each vertex is replaced by the arithmetic mean of the two numbers appearing at the adjacent vertices. If this procedure is repeated a total of 2006 times, then the resulting number at A can be expressed as m/n, where m and n are relatively prime positive integers. Compute the remainder obtained when $m+n$ is divided by 2006.

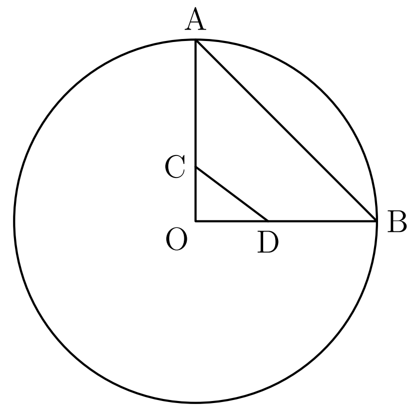
– Acute triangle ABC satisfies $AB = 2AC$ and $AB^4 + BC^4 + CA^4 = 2006 \cdot 10^{10}$. Tetrahedron $DEFP$ is formed by choosing points D, E, and F on the segments BC, CA, and AB (respectively) and folding A, B, C, over EF, FD, and DE (respectively) to the common point P. Let R denote the circumradius of $DEFP$. Compute the smallest positive integer N for which we can be certain that $n \geq R$. It may be helpful to use $\sqrt[4]{1239} = 5.9329109\ldots$.

– **Ultimate Question**

– Find the real number x such that

$$\sqrt{x-9}+\sqrt{x-6}=\sqrt{x-1}.$$

– Let $T = TNFTPP$. Points A and B lie on a circle centered at O such that $\angle AOB$ is right. Points C and D lie on radii OA and OB respectively such that $AC = T-3$, $CD = 5$, and $BD = 6$. Determine the area of quadrilateral $ACDB$.



[b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 10$.

– Let $T = TNFTPP$. When properly sorted, $T - 35$ math books on a shelf are arranged in alphabetical order from left to right. An eager student checked out and read all of them. Unfortunately, the student did not realize how the books were sorted, and so after finishing the student put the books back on the shelf in a random order. If all arrangements are equally likely, the probability that exactly 6 of the books were returned to their correct (original) position can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

  [b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 44$.

– Let $T = TNFTPP$. As n ranges over the integers, the expression $n^4 - 898n^2 + T - 2160$ evaluates to just one prime number. Find this prime.

  [b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 2161$.

– Let $T = TNFTPP$, and let S be the sum of the digits of T. In triangle ABC, points D, E, and F are the feet of the angle bisectors of $\angle A$, $\angle B$, $\angle C$ respectively. Let point P be the intersection of segments AD and BE, and let p denote the perimeter of ABC. If $AP = 3PD$, $BE = S - 1$, and $CF = 9$, then the value of $\frac{AD}{p}$ can be expressed uniquely as $\frac{\sqrt{m}}{n}$ where m and n are positive integers such that m is not divisible by the square of any prime. Find $m + n$.

  [b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 1801$.

– Let $T = TNFTPP$. x and y are nonzero real numbers such that

  $$18x - 4x^2 + 2x^3 - 9y - 10xy - x^2y + Ty^2 + 2xy^2 - y^3 = 0.$$

  The smallest possible value of $\frac{y}{x}$ is equal to $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

  [b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 6$.

– Let $T = TNFTPP$. Triangle ABC has integer side lengths, including $BC = 100T - 4$, and a right angle, $\angle ABC$. Let r and s denote the inradius and semiperimeter of ABC respectively. Find the "perimeter" of the triangle ABC which minimizes $\frac{s}{r}$.

  [b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 7$

– Let $T = TNFTPP$, and let S be the sum of the digits of T. Cyclic quadrilateral $ABCD$ has side lengths $AB = S - 11$, $BC = 2$, $CD = 3$, and $DA = 10$. Let M and N be the midpoints of sides AD and BC. The diagonals AC and BD intersect MN at P and Q respectively. $\frac{PQ}{MN}$ can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

[b]Note: This is part of the Ultimate Problem, where each question depended on the previous question. For those who wanted to try the problem separately, $T = 2378$