2019 China National Olympiad

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– Day 1

1 Let $a,b,c,d,e \geq -1$ and $a+b+c+d+e=5$. Find the maximum and minimum value of $S=(a+b)(b+c)(c+d)(d+e)(e+a)$.

2 Call a set of 3 positive integers $\{a,b,c\}$ a *Pythagorean* set if a,b,c are the lengths of the 3 sides of a right-angled triangle. Prove that for any 2 Pythagorean sets P,Q, there exists a positive integer $m\geq 2$ and Pythagorean sets $P_1,P_2,\ldots,P_m$ such that $P=P_1, Q=P_m$, and $\forall 1\leq i\leq m-1$, $P_i\cap P_{i+1}\neq\emptyset$.

3 Let O be the circumcenter of $\triangle ABC(AB<AC)$, and D be a point on the internal angle bisector of $\angle BAC$. Point E lies on BC, satisfying $OE \parallel AD$, $DE\perp BC$. Point K lies on EB extended such that $EK=EA$. The circumcircle of $\triangle ADK$ meets BC at $P\neq K$, and meets the circumcircle of $\triangle ABC$ at $Q\neq A$. Prove that PQ is tangent to the circumcircle of $\triangle ABC$.

– Day 2

4 Given an ellipse that is not a circle.
(1) Prove that the rhombus tangent to the ellipse at all four of its sides with minimum area is unique.
(2) Construct this rhombus using a compass and a straight edge.

5 Given is an $n\times n$ board, with an integer written in each grid. For each move, I can choose any grid, and add 1 to all $2n-1$ numbers in its row and column. Find the largest $N(n)$, such that for any initial choice of integers, I can make a finite number of moves so that there are at least $N(n)$ even numbers on the board.

6 The point $P_1, P_2,\cdots, P_{2018}$ is placed inside or on the boundary of a given regular pentagon. Find all placement methods are made so that

$$S=\sum_{1\leq i<j\leq 2018}|P_iP_j|^2$$

takes the maximum value.