Rioplatense Mathematical Olympiad, Level 3 2018

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by parmenides51

– Day 1

1 Determine if there are 2018 different positive integers such that the sum of their squares is a perfect cube and the sum of their cubes is a perfect square.

2 Let P be a point outside a circumference Γ, and let PA be one of the tangents from P to Γ. Line l passes through P and intersects Γ at B and C, with B between P and C. Let D be the symmetric of B with respect to P. Let ω_1 and ω_2 be the circles circumscribed to the triangles DAC and PAB respectively. ω_1 and ω_2 intersect at $E \neq A$. Line EB cuts back to ω_1 in F. Prove that $CF = AB$.

3 Determine all the triples $\{a, b, c\}$ of positive integers coprime (not necessarily pairwise prime) such that $a+b+c$ simultaneously divides the three numbers $a^{12}+b^{12}+c^{12}$, $a^{23}+b^{23}+c^{23}$ and $a^{11004}+b^{11004}+c^{11004}$

– Day 2

4 Let ABC be an acute triangle with $AC > AB$. be Γ the circumcircle circumscribed to the triangle ABC and D the midpoint of the smallest arc BC of this circle. Let E and F points of the segments AB and AC respectively such that $AE = AF$. Let $P \neq A$ be the second intersection point of the circumcircle circumscribed to AEF with Γ. Let G and H be the intersections of lines PE and PF with Γ other than P, respectively. Let J and K be the intersection points of lines DG and DH with lines AB and AC respectively. Show that the JK line passes through the midpoint of BC

5 Let n be a positive integer. Find all n- rows $(a_1, a_2, ..., a_n)$ of different positive integers such that

$$\frac{(a_1+d)(a_2+d)\cdots(a_n+d)}{a_1a_2\cdots a_n}$$

is integer for every integer $d \geq 0$

6 A company has n employees. It is known that each of the employees works at least one of the 7 days of the week, with the exception of an employee who does not work any of the 7 days. Furthermore, for any two of these n employees, there are at least 3 days of the week in which one of the two works that day and the other does not (it is not necessarily the same employee who works those days). Determine the highest possible value of n.