www.artofproblemsolving.com/community/c789645
by djmathman, grupyorum

– Day 1

1 Find all pairs (x, y) of real numbers that satisfy,

$$x^2 + y^2 + x + y = xy(x+y) - \frac{10}{27}$$
$$|xy| \leq \frac{25}{9}.$$

2 Let P be a point in the interior of the triangle ABC. The lines AP, BP, and CP intersect the sides BC, CA, and AB at D, E, and F, respectively. A point Q is taken on the ray $[BE$ such that $E \in [BQ]$ and $m(\widehat{EDQ}) = m(\widehat{BDF})$. If BE and AD are perpendicular, and $|DQ| = 2|BD|$, prove that $m(\widehat{FDE}) = 60^\circ$.

3 A sequence $a_1, a_2, \ldots$ satisfy

$$\sum_{i=1}^{n} a_{\lfloor \frac{n}{i} \rfloor} = n^{10},$$

for every $n \in \mathbb{N}$.
Let c be a positive integer. Prove that, for every positive integer n,

$$\frac{c^{a_n} - c^{a_{n-1}}}{n}$$

is an integer.

– Day 2

4 In a triangle ABC, the bisector of the angle A intersects the excircle that is tangential to side $[BC]$ at two points D and E such that $D \in [AE]$. Prove that,

$$\frac{|AD|}{|AE|} \leq \frac{|BC|^2}{|DE|^2}.$$

5 Let a_1, a_2, a_3, a_4 be positive integers, with the property that it is impossible to assign them around a circle where all the neighbors are coprime. Let $i, j, k \in \{1, 2, 3, 4\}$ with $i \neq j$, $j \neq k$, and $k \neq i$. Determine the maximum number of triples (i, j, k) for which

$$(\gcd(a_i, a_j))^2 | a_k.$$

6 Initially, there are 2018 distinct boxes on a table. In the first stage, Yazan and Bozan, starting with Yazan, take turns make 2016 moves each, such that, in each move, the person whose turn selects a pair of boxes that is not written on the board, and writes the pair on the board.

In the second stage, Bozan enumerates the 4032 pairs with numbers from $1, 2, \ldots, 4032$, in whichever order he wants, and puts k balls in each boxes written contained in the k^{th} pair. Is there a strategy for Bozan that guarantees that the number of balls in each box are distinct?