Cono Sur Olympiad 2015

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–	**Day 1**
1	Show that, for any integer n, the number $n^3 - 9n + 27$ is not divisible by 81.
2	$3n$ lines are drawn on the plane ($n > 1$), such that no two of them are parallel and no three of them are concurrent. Prove that, if $2n$ of the lines are coloured red and the other n lines blue, there are at least two regions of the plane such that all of their borders are red. Note: for each region, all of its borders are contained in the original set of lines, and no line passes through the region.
3	Given a acute triangle PA_1B_1 is inscribed in the circle Γ with radius 1. for all integers $n \geq 1$ are defined: C_n the foot of the perpendicular from P to A_nB_n O_n is the center of $\odot(PA_nB_n)$ A_{n+1} is the foot of the perpendicular from C_n to PA_n $B_{n+1} \equiv PB_n \cap O_nA_{n+1}$ If $PC_1 = \sqrt{2}$, find the length of PO_{2015} Cono Sur Olympiad - 2015 - Day 1 - Problem 3
–	**Day 2**
4	Let $ABCD$ be a convex quadrilateral such that $\angle BAD = 90^\circ$ and its diagonals AC and BD are perpendicular. Let M be the midpoint of side CD, and E be the intersection of BM and AC. Let F be a point on side AD such that BM and EF are perpendicular. If $CE = AF\sqrt{2}$ and $FD = CE\sqrt{2}$, show that $ABCD$ is a square.
5	Determine if there exists an infinite sequence of not necessarily distinct positive integers $a_1, a_2, a_3, \dots$ such that for any positive integers m and n where $1 \leq m < n$, the number $a_{m+1} + a_{m+2} + \dots + a_n$ is not divisible by $a_1 + a_2 + \dots + a_m$.
6	Let $S = \{1, 2, 3, \dots, 2046, 2047, 2048\}$. Two subsets A and B of S are said to be *friends* if the following conditions are true: - They do not share any elements. - They both have the same number of elements. - The product of all elements from A equals the product of all elements from B.

Prove that there are two subsets of S that are *friends* such that each one of them contains at least 738 elements.