Serbia Additional Team Selection Test 2013

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by mihajlon

1 We call polynomials $A(x) = a_nx^n + ... + a_1x + a_0$ and $B(x) = b_mx^m + ... + b_1x + b_0$ ($a_nb_m \neq 0$) similar if the following conditions hold: (i) $n = m$; (ii) There is a permutation π of the set $\{0, 1, ..., n\}$ such that $b_i = a_{\pi(i)}$ for each $i \in 0, 1, ..., n$.
Let $P(x)$ and $Q(x)$ be similar polynomials with integer coefficients. Given that $P(16) = 3^{2012}$, find the smallest possible value of $|Q(3^{2012})|$.

Proposed by Milos Milosavljevic

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2 In an acute $\triangle ABC$ ($AB \neq AC$) with angle α at the vertex A, point E is the nine-point center, and P a point on the segment AE. If $\angle ABP = \angle ACP = x$, prove that $x = 90 - 2\alpha$.

Proposed by Dusan Djukic

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3 Let $p > 3$ be a given prime number. For a set $S \subseteq \mathbb{Z}$ and $a \in \mathbb{N}$, define $S_a = \{x \in \{0, 1, 2, ..., p-1\}$ – $(\exists_s \in S) x \equiv_p a \cdot s\}$. (a) How many sets $S \subseteq \{1, 2, ..., p-1\}$ are there for which the sequence $S_1, S_2, ..., S_{p-1}$ contains exactly two distinct terms? (b) Determine all numbers $k \in \mathbb{N}$ for which there is a set $S \subseteq \{1, 2, ..., p-1\}$ such that the sequence $S_1, S_2, ..., S_{p-1}$ contains exactly k distinct terms.

Proposed by Milan Basic and Milos Milosavljevic
