European Mathematical Cup 2018

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– Junior Division

1 Let a,b,c be non-zero real numbers such that $a^2+b+c=\frac{1}{a},b^2+c+a=\frac{1}{b},c^2+a+b=\frac{1}{c}.$ Prove that at least two of a,b,c are equal.

2 Find all pairs $(x;y)$ of positive integers such that

$$xy|x^2+2y-1.$$

3 Let ABC be an acute triangle with $|AB|<|AC|$and orthocenter H. The circle with center A and radius$|AC|$ intersects the circumcircle of $\triangle ABC$ at point D and the circle with center A and radius$|AB|$ intersects the segment $\overline{AD}$ at point K. The line through K parallel to CD intersects BC at the point L. If M is the midpoint of $\overline{BC}$ and N is the foot of the perpendicular from H to $AL,$ prove that the line MN bisects the segment $\overline{AH}$.

4 Let n be a positive integer. Ana and Banana are playing the following game:
First, Ana arranges $2n$ cups in a row on a table, each facing upside-down. She then places a ball under a cup
and makes a hole in the table under some other cup. Banana then gives a finite sequence of commands to Ana,
where each command consists of swapping two adjacent cups in the row.
Her goal is to achieve that the ball has fallen into the hole during the game. Assuming Banana has no information
about the position of the hole and the position of the ball at any point, what is the smallest number of commands
she has to give in order to achieve her goal?

– Senior Division

1 A partition of a positive integer is even if all its elements are even numbers. Similarly, a partition is odd if all its elements are odd. Determine all positive integers n such that the number of even partitions of n is equal to the number of odd partitions of $n.$
Remark: A partition of a positive integer n is a non-decreasing sequence of positive integers whose sum of
elements equals n. For example, $(2;3;4),(1;2;2;2;2)$ and (9) are partitions of $9.$

2 Let ABC be a triangle with$|AB| < |AC|$. Let k be the circumcircle of $\triangle ABC$ and let O be the center of k. Point M is the midpoint of the arc BC of k not containing A. Let D be the second intersection of the perpendicular line from M to AB with k and E be the second intersection of the perpendicular line from M to AC with k. Points $X$and Y are the intersections of CD and BE with OM respectively. Denote by k_b and k_c circumcircles of triangles BDX and CEY respectively. Let G and H be the second intersections of k_b and k_c with AB and AC respectively. Denote by ka the circumcircle of triangle $AGH.$
Prove that O is the circumcenter of $\triangle O_aO_bO_c$,where O_a, O_b, O_c are the centers of k_a, k_b, k_c respectively.

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3 For which real numbers $k > 1$ does there exist a bounded set of positive real numbers S with at
least 3 elements such that

$$k(a-b) \in S$$

for all $a, b \in S$ with $a > b?$
Remark: A set of positive real numbers S is bounded if there exists a positive real number M such that $x < M$ for all $x \in S.$

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4 Let $x; y; m; n$ be integers greater than 1 such that
