Purple Comet Problems 2016

www.artofproblemsolving.com/community/c831722

by Snakes, anser, techguy2

–	High School
1	Two integers have a sum of 2016 and a difference of 500. Find the larger of the two integers.
2	The trapezoid below has bases with lengths 7 and 17 and area 120. Find the difference of the areas of the two triangles. `https://i.snag.gy/BlqcSQ.jpg`
3	Find the positive integer n such that 10^n cubic centimeters is the same as 1 cubic kilometer.
4	One side of a rectangle has length 18. The area plus the perimeter of the rectangle is 2016. Find the perimeter of the rectangle.
5	Julius has a set of five positive integers whose mean is 100. If Julius removes the median of the set of five numbers, the mean of the set increases by 5, and the median of the set decreases by 5. Find the maximum possible value of the largest of the five numbers Julius has.
6	The following diagram shows a square where each side has seven dots that divide the side into six equal segments. All the line segments that connect these dots that form a 45° angle with a side of the square are drawn as shown. The area of the shaded region is 75. Find the area of the original square. `https://i.snag.gy/Jzx9Fn.jpg`
7	Positive integers m and n are both greater than 50, have a least common multiple equal to 480, and have a greatest common divisor equal to 12. Find m + n.
8	The map below shows an east/west road connecting the towns of Acorn, Centerville, and Midland, and a north/south road from Centerville to Drake. The distances from Acorn to Centerville, from Centerville to Midland, and from Centerville to Drake are each 60 kilometers. At noon Aaron starts at Acorn and bicycles east at 17 kilometers per hour, Michael starts at Midland and bicycles west at 7 kilometers per hour, and David starts at Drake and bicycles at a constant rate in a straight line across an open field. All three bicyclists arrive at exactly the same

time at a point along the road from Centerville to Midland. Find the number of kilometers that David bicycles.

https://i.snag.gy/Ik094i.jpg

---

9 Find the sum of all perfect squares that divide 2016.

---

10 Jeremy wrote all the three-digit integers from 100 to 999 on a blackboard. Then Allison erased each of the 2700 digits Jeremy wrote and replaced each digit with the square of that digit. Thus, Allison replaced every 1 with a 1, every 2 with a 4, every 3 with a 9, every 4 with a 16, and so forth. The proportion of all the digits Allison wrote that were ones is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

---

11 Find the number of three-digit positive integers which have three distinct digits where the sum of the digits is an even number such as 925 and 824.

---

12 Find the number whose reciprocal is the sum of the reciprocal of $9 + 15i$ and the reciprocal of $9 - 15i$.

---

13 In △ABC shown below, AB = AC, AF = EF, and EH = CH = DH = GH = DG = BG. Also, CHE = F GH. Find the degree measure of BAC.

https://i.snag.gy/ZyxQVX.jpg

---

14 Find the greatest possible value of $pq + r$, where p, q, and r are (not necessarily distinct) prime numbers satisfying $pq + qr + rp = 2016$.

---

15 Find the least positive integer of the form a b a a b a, where a and b are distinct digits, such that the integer can be written as a product of six distinct primes

---

16 Henry rolls a fair die. If the die shows the number k, Henry will then roll the die k more times. The probability that Henry will never roll a 3 or a 6 either on his first roll or on one of the k subsequent rolls is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

---

17 The cubic polynomials $p(x)$ and $q(x)$ satisfy
$p(1) = q(2)$
$p(3) = q(4)$
$p(5) = q(6)$
$p(7) = q(8) + 13.$
Find $p(9) - q(10)$.

---

18 The Tasty Candy Company always puts the same number of pieces of candy into each one-pound bag of candy they sell. Mike bought 4 one-pound bags and gave each person in his class 15 pieces of candy. Mike had 23 pieces of candy left over. Betsy bought 5 one-pound bags and gave 23 pieces of candy to each teacher in her school. Betsy had 15 pieces of candy left over. Find the least number of pieces of candy the Tasty Candy Company could have placed in each one-pound bag.

---

19 Jar#1 contains five red marbles, three blue marbles, and one green marble.
Jar#2 contains five blue marbles, three green marbles, and one red marble.
Jar#3 contains five green marbles, three red marbles, and one blue marble.
You randomly select one marble from each jar. Given that you select one marble of each color, the probability that the red marble came from jar#1, the blue marble came from jar#2, and the green marble came from jar#3 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Find m + n.

---

20 Positive integers a, b, c, d, and e satisfy the equations

$$(a+1)(3bc+1) = d+3e+1$$

$$(b+1)(3ca+1) = 3d+e+13$$

$$(c+1)(3ab+1) = 4(26-d-e)-1$$

Find $d^2 + e^2$.

---

21 On equilateral $\triangle ABC$ point D lies on BC a distance 1 from B, point E lies on AC a distance 1 from C, and point F lies on AB a distance 1 from A. Segments AD, BE, and CF intersect in pairs at points G, H, and J which are the vertices of another equilateral triangle. The area of $\triangle ABC$ is twice the area of $\triangle GHJ$. The side length of $\triangle ABC$ can be written $\frac{r+\sqrt{s}}{t}$, where r, s, and t are relatively prime positive integers. Find $r+s+t$.

`https://i.snag.gy/TKU5Fc.jpg`

---

22 In $\triangle ABC$, $cos\angle A = \frac{2}{3}$, $cos\angle B = \frac{1}{9}$, and $BC = 24$. Find the length AC.

---

23 Sixteen dots are arranged in a four by four grid as shown. The distance between any two dots in the grid is the minimum number of horizontal and vertical steps along the grid lines it takes to get from one dot to the other. For example, two adjacent dots are a distance 1 apart, and two dots at opposite corners of the grid are a distance 6 apart. The mean distance between two distinct dots in the grid is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

`https://i.snag.gy/c1tB7z.jpg`

---

24 Find the largest prime p such that p divides $2^{p+1} + 3^{p+1} + 5^{p+1} + 7^{p+1}$.

---

25 For n measured in degrees, let $T(n) = \cos^2(30^\circ - n) - \cos(30^\circ - n)\cos(30^\circ + n) + \cos^2(30^\circ + n)$. Evaluate

$$4\sum_{n=1}^{30} n \cdot T(n).$$

---

26 Find the sum of all values of a such that there are positive integers a and b satisfying $(a - b)\sqrt{ab} = 2016$.

---

27 A container the shape of a pyramid has a 12 12 square base, and the other four edges each have length 11. The container is partially filled with liquid so that when one of its triangular faces is lying on a flat surface, the level of the liquid is half the distance from the surface to the top edge of the container. Find the volume of the liquid in the container.

https://snag.gy/CdvpUq.jpg

---

28 Find the sum of all the possible values of xy such that x and y are positive integers satisfying $(x^2 + 1)(y^2 + 1) + 2(x - y)(1 - xy) = 4(1 + xy) + 140$.

---

29 Ten square tiles are placed in a row, and each can be painted with one of the four colors red (R), yellow (Y), blue (B), and white (W). Find the number of ways this can be done so that each block of five adjacent tiles contains at least one tile of each color. That is, count the patterns RWBWYRRBWY and WWBYRWYBWR but not RWBYYBWWRY because the five adjacent tiles colored BYYBW does not include the color red.

---

30 Some identically sized spheres are piled in n layers in the shape of a square pyramid with one sphere in the top layer, 4 spheres in the second layer, 9 spheres in the third layer, and so forth so that the bottom layer has a square array of n^2 spheres. In each layer the centers of the spheres form a square grid so that each sphere is tangent to any sphere adjacent to it on the grid. Each sphere in an upper level is tangent to the four spheres directly below it. The diagram shows how the first three layers of spheres are stacked. A square pyramid is built around the pile of spheres so that the sides of the pyramid are tangent to the spheres on the outside of the pile. There is a positive integer m such that as n gets large, the ratio of the volume of the pyramid to the total volume inside all of the spheres approaches $\frac{\sqrt{m}}{\pi}$. Find m.

https://snag.gy/bIwyl6.jpg

– Middle School

---

1 Mike has 12 books, Sean has 9 books, and little Sherry has only 4 books. Find the percentage of these books that Sean has.

---

2 The figure below was formed by taking four squares, each with side length 5, and putting one on each side of a square with side length 20. Find the perimeter of the figure below.

https://snag.gy/LGimC8.jpg

---

3 The sum of the numbers $3a-4$, $3b-4$, and $3c-4$ is 2016. Find the sum of the numbers $4a-3$, $4b-3$, and $4c-3$.

---

4 The following diagram shows a square where each side has four dots that divide the side into three equal segments. The shaded region has area 105. Find the area of the original square.

https://snag.gy/r60Y7k.jpg

---

5 A 2 meter long bookshelf is filled end-to-end with 46 books. Some of the books are 3 centimeters thick while all the others are 5 centimeters thick. Find the number of books on the shelf that are 3 centimeters thick.

---

6 Find the number of three-digit positive integers where the digits are three different prime numbers. For example, count 235 but not 553.

---

7 Melanie has $4\frac{2}{5}$ cups of flour. The recipe for one batch of cookies calls for $1\frac{1}{2}$ cups of flour. Melanie plans to make $2\frac{1}{2}$ batches of cookies. When she is done, she will have $\frac{m}{n}$ cups of flour remaining, where m and n are relatively prime positive integers. Find $m+n$.

---

8 The figure below has a 1 1 square, a 2 2 square, a 3 3 square, a 4 4 square, and a 5 5 square. Each of the larger squares shares a corner with the 1 1 square. Find the area of the region covered by these five squares.

https://snag.gy/1AfJWt.jpg

---

9 Find the value of x such that $2^{x+3}-2^{x-3}=2016.$

---

10 Mildred the cow is tied with a rope to the side of a square shed with side length 10 meters. The rope is attached to the shed at a point two meters from one corner of the shed. The rope is 14 meters long. The area of grass growing around the shed that Mildred can reach is given by $n\pi$ square meters, where n is a positive integer. Find n.

---

11 One evening a theater sold 300 tickets for a concert. Each ticket sold for $40, and all tickets were purchased using $5, $10, and $20 bills. At the end of the evening the theater had received twice as many $10 bills as $20 bills, and 20 more $5 bills than $10 bills. How many bills did the theater receive altogether?

---

12 Find the number of squares such that the sides of the square are segments in the following diagram and where the dot is inside the square.

https://snag.gy/qXBIY4.jpg

---

13 One afternoon Elizabeth noticed that twice as many cars on the expressway carried only a driver as compared to the number of cars that carried a driver and one passenger. She also noted that twice as many cars carried a driver and one passenger as those that carried a driver and two passengers. Only 10

---

14 Find the number of positive integers n such that a regular polygon with n sides has internal angles with measures equal to an integer number of degrees.

---

15 The real numbers x, y, and z satisfy the system of equations

$$x^2 + 27 = -8y + 10z$$

$$y^2 + 196 = 18z + 13x$$

$$z^2 + 119 = -3x + 30y$$

Find $x + 3y + 5z$.

---

16 The figure below shows a barn in the shape of two congruent pentagonal prisms that intersect at right angles and have a common center. The ends of the prisms are made of a 12 foot by 7 foot rectangle surmounted by an isosceles triangle with sides 10 feet, 10 feet, and 12 feet. Each prism is 30 feet long. Find the volume of the barn in cubic feet.

`https://snag.gy/Ox9CUp.jpg`

---

17 Suzie flips a fair coin 6 times. The probability that Suzie flips 3 heads in a row but not 4 heads in a row is given by $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

---

18 Find the least positive integer N that is 50 times the number of positive integer divisors that N has.

---

19 Find the positive integer n such that the least common multiple of n and $n - 30$ is $n + 1320$.

---

20 The 24 unshaded squares in the 5 5 grid below can be tiled with twelve 1 2 tiles. One such tiling is shown. Find the number of ways the grid can be tiled.

`https://snag.gy/KMoPrF.jpg`
