**India National Olympiad 2019**

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by div5252

**1** Let $ABC$ be a triangle with $\angle BAC > 90$. Let $D$ be a point on the segment $BC$ and $E$ be a point on line $AD$ such that $AB$ is tangent to the circumcircle of triangle $ACD$ at $A$ and $BE$ is perpendicular to $AD$. Given that $CA = CD$ and $AE = CE$. Determine $\angle BCA$ in degrees.

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**2** Let $A_1B_1C_1D_1E_1$ be a regular pentagon.For $2 \le n \le 11$, let $A_nB_nC_nD_nE_n$ be the pentagon whose vertices are the midpoint of the sides $A_{n-1}B_{n-1}C_{n-1}D_{n-1}E_{n-1}$. All the $5$ vertices of each of the $11$ pentagons are arbitrarily coloured red or blue. Prove that four points among these $55$ points have the same colour and form the vertices of a cyclic quadrilateral.

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**3** Let $m, n$ be distinct positive integers. Prove that

$$gcd(m, n) + gcd(m + 1, n + 1) + gcd(m + 2, n + 2) \le 2|m - n| + 1.$$

Further, determine when equality holds.

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**4** Let $n$ and $M$ be positive integers such that $M > n^{n-1}$. Prove that there are $n$ distinct primes $p_1, p_2, p_3 \cdots, p_n$ such that $p_j$ divides $M + j$ for all $1 \le j \le n$.

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**5** Let $AB$ be the diameter of a circle $\Gamma$ and let $C$ be a point on $\Gamma$ different from $A$ and $B$. Let $D$ be the foot of perpendicular from $C$ on to $AB$.Let $K$ be a point on the segment $CD$ such that $AC$ is equal to the semi perimeter of $ADK$.Show that the excircle of $ADK$ opposite $A$ is tangent to $\Gamma$.

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**6** Let $f$ be a function defined from $((x, y) : x, y$ real, $xy \neq 0)$ to the set of all positive real numbers such that $(i) f(xy, z) = f(x, z) \cdot f(y, z)$ for all $x, y \neq 0$ $(ii) f(x, yz) = f(x, y) \cdot f(x, z)$ for all $x, y \neq 0$ $(iii) f(x, 1 - x) = 1$ for all $x \neq 0, 1$
Prove that $(a) f(x, x) = f(x, -x) = 1$ for all $x \neq 0$ $(b) f(x, y) \cdot f(y, x) = 1$ for all $x, y \neq 0$

The condition (ii) was left out in the paper leading to an incomplete problem during contest.

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