**Harvard-MIT Mathematics Tournament 2019**

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by CantonMathGuy, mcyoder, rd123

– Team Round

**1** Let $ABCD$ be a parallelogram. Points $X$ and $Y$ lie on segments $AB$ and $AD$ respectively, and $AC$ intersects $XY$ at point $Z$. Prove that

$$\frac{AB}{AX}+\frac{AD}{AY}=\frac{AC}{AZ}.$$

**2** Let $\mathbb{N}=\{1,2,3,\dots\}$ be the set of all positive integers, and let $f$ be a bijection from $\mathbb{N}$ to $\mathbb{N}$. Must there exist some positive integer $n$ such that $(f(1),f(2),\dots,f(n))$ is a permutation of $(1,2,\dots,n)$?

**3** For any angle $0<\theta<\pi/2$, show that

$$0<\sin\theta+\cos\theta+\tan\theta+\cot\theta-\sec\theta-\csc\theta<1.$$

**4** Find all positive integers $n$ for which there do not exist $n$ consecutive composite positive integers less than $n!$.

**5** Find all positive integers $n$ such that the unit segments of an $n\times n$ grid of unit squares can be partitioned into groups of three such that the segments of each group share a common vertex.

**6** Scalene triangle $ABC$ satisfies $\angle A=60^\circ$. Let the circumcenter of $ABC$ be $O$, the orthocenter be $H$, and the incenter be $I$. Let $D,T$ be the points where line $BC$ intersects the internal and external angle bisectors of $\angle A$, respectively. Choose point $X$ on the circumcircle of $\triangle IHO$ such that $HX\parallel AI$. Prove that $OD\perp TX$.

**7** A convex polygon on the plane is called *wide* if the projection of the polygon onto any line in the same plane is a segment with length at least 1. Prove that a circle of radius $\frac{1}{3}$ can be placed completely inside any wide polygon.

**8** Can the set of lattice points $\{(x,y)\mid x,y\in\mathbb{Z},1\le x,y\le 252,x\ne y\}$ be colored using 10 distinct colors such that for all $a\neq b$, $b\neq c$, the colors of $(a,b)$ and $(b,c)$ are distinct?

 

**9** Let $p > 2$ be a prime number. $\mathbb{F}_p[x]$ is defined as the set of polynomials in $x$ with coefficients in $\mathbb{F}_p$ (the integers modulo $p$ with usual addition and subtraction), so that two polynomials are equal if and only if the coefficients of $x^k$ are equal in $\mathbb{F}_p$ for each nonnegative integer $k$. For example, $(x+2)(2x+3) = 2x^2+2x+1$ in $\mathbb{F}_5[x]$ because the corresponding coefficients are equal modulo 5.

Let $f, g \in \mathbb{F}_p[x]$. The pair $(f, g)$ is called *compositional* if

$$f(g(x)) \equiv x^{p^2} - x$$

in $\mathbb{F}_p[x]$. Find, with proof, the number of compositional pairs.

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**10** Prove that for all positive integers $n$, all complex roots $r$ of the polynomial

$$P(x) = (2n)x^{2n} + (2n-1)x^{2n-1} + \cdots + (n+1)x^{n+1} + nx^n + (n+1)x^{n-1} + \cdots + (2n-1)x + 2n$$

lie on the unit circle (i.e. $|r| = 1$).

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**–** Algebra and Number Theory

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**1** What is the smallest positive integer that cannot be written as the sum of two nonnegative palindromic integers? (An integer is *palindromic* if the sequence of decimal digits are the same when read backwards.)

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**2** Let $N = 2^{(2^2)}$ and $x$ be a real number such that $N^{(N^N)} = 2^{(2^x)}$. Find $x$.

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**3** Let $x$ and $y$ be positive real numbers. Define $a = 1 + \frac{x}{y}$ and $b = 1 + \frac{y}{x}$. If $a^2 + b^2 = 15$, compute $a^3 + b^3$.

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**4** Let $\mathbb{N}$ be the set of positive integers, and let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

- $f(1) = 1$,
- for $n \in \mathbb{N}$, $f(2n) = 2f(n)$ and $f(2n+1) = 2f(n) - 1$.

Determine the sum of all positive integer solutions to $f(x) = 19$ that do not exceed 2019.

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**5** Let $a_1, a_2, \ldots$ be an arithmetic sequence and $b_1, b_2, \ldots$ be a geometric sequence. Suppose that $a_1 b_1 = 20$, $a_2 b_2 = 19$, and $a_3 b_3 = 14$. Find the greatest possible value of $a_4 b_4$.

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**6** For positive reals $p$ and $q$, define the *remainder* when $p$ and $q$ as the smallest nonnegative real $r$ such that $\frac{p-r}{q}$ is an integer. For an ordered pair $(a, b)$ of positive integers, let $r_1$ and $r_2$ be the remainder when $a\sqrt{2} + b\sqrt{3}$ is divided by $\sqrt{2}$ and $\sqrt{3}$ respectively. Find the number of pairs $(a, b)$ such that $a, b \le 20$ and $r_1 + r_2 = \sqrt{2}$.

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**7** Find the value of

$$\sum_{a=1}^{\infty}\sum_{b=1}^{\infty}\sum_{c=1}^{\infty}\frac{ab(3a+c)}{4^{a+b+c}(a+b)(b+c)(c+a)}.$$

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**8** There is a unique function $f:\mathbb{N}\to\mathbb{R}$ such that $f(1)>0$ and such that

$$\sum_{d|n} f(d)f\left(\frac{n}{d}\right)=1$$

for all $n\geq 1$. What is $f(2018^{2019})$?

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**9** Tessa the hyper-ant has a 2019-dimensional hypercube. For a real number $k$, she calls a placement of nonzero real numbers on the $2^{2019}$ vertices of the hypercube [i]$k$-harmonic[/i] if for any vertex, the sum of all 2019 numbers that are edge-adjacent to this vertex is equal to $k$ times the number on this vertex. Let $S$ be the set of all possible values of $k$ such that there exists a $k$-harmonic placement. Find $\sum_{k\in S}|k|$.

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**10** The sequence of integers $\{a_i\}_{i=0}^{\infty}$ satisfies $a_0=3$, $a_1=4$, and

$$a_{n+2}=a_{n+1}a_n+\left\lceil\sqrt{a_{n+1}^2-1}\sqrt{a_n^2-1}\right\rceil$$

for $n\geq 0$. Evaluate the sum

$$\sum_{n=0}^{\infty}\left(\frac{a_{n+3}}{a_{n+2}}-\frac{a_{n+2}}{a_n}+\frac{a_{n+1}}{a_{n+3}}-\frac{a_n}{a_{n+1}}\right).$$

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**–** Combinatorics

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**1** How many distinct permutations of the letters in the word REDDER are there that do not contain a palindromic substring of length at least two? (A *substring* is a continuous block of letters that is part of the string. A string is *palindromic* if it is the same when read backwards.)

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**2** Your math friend Steven rolls five fair icosahedral dice (each of which is labelled $1,2,\dots,20$ on its sides). He conceals the results but tells you that at least half the rolls are $20$. Suspicious, you examine the first two dice and find that they show $20$ and $19$ in that order. Assuming that Steven is truthful, what is the probability that all three remaining concealed dice show $20$?

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**3** Reimu and Sanae play a game using $4$ fair coins. Initially both sides of each coin are white. Starting with Reimu, they take turns to color one of the white sides either red or green. After all sides are colored, the four coins are tossed. If there are more red sides showing up, then Reimu wins, and if there are more green sides showing up, then Sanae wins. However, if there is an equal number of red sides and green sides, then *neither* of them wins. Given that both of them play optimally to maximize the probability of winning, what is the probability that Reimu wins?

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**4** Yannick is playing a game with $100$ rounds, starting with $1$ coin. During each round, there is an $n\%$ chance that he gains an extra coin, where $n$ is the number of coins he has at the beginning of the round. What is the expected number of coins he will have at the end of the game?

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**5** Contessa is taking a random lattice walk in the plane, starting at $(1,1)$. (In a random lattice walk, one moves up, down, left, or right $1$ unit with equal probability at each step.) If she lands on a point of the form $(6m,6n)$ for $m,n\in\mathbb{Z}$, she ascends to heaven, but if she lands on a point of the form $(6m+3,6n+3)$ for $m,n\in\mathbb{Z}$, she descends to hell. What is the probability she ascends to heaven?

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**6** A point $P$ lies at the center of square $ABCD$. A sequence of points $\{P_n\}$ is determined by $P_0=P$, and given point $P_i$, point $P_{i+1}$ is obtained by reflecting $P_i$ over one of the four lines $AB$, $BC$, $CD$, $DA$, chosen uniformly at random and independently for each $i$. What is the probability that $P_8=P$?

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**7** In an election for the Peer Pressure High School student council president, there are 2019 voters and two candidates Alice and Celia (who are voters themselves). At the beginning, Alice and Celia both vote for themselves, and Alice's boyfriend Bob votes for Alice as well. Then one by one, each of the remaining 2016 voters votes for a candidate randomly, with probabilities proportional to the current number of the respective candidate's votes. For example, the first undecided voter David has a $\frac{2}{3}$ probability of voting for Alice and a $\frac{1}{3}$ probability of voting for Celia.

What is the probability that Alice wins the election (by having more votes than Celia)?

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**8** For a positive integer $N$, we color the positive divisors of $N$ (including 1 and $N$) with four colors. A coloring is called *multichromatic* if whenever $a$, $b$ and $\gcd(a,b)$ are pairwise distinct divisors of $N$, then they have pairwise distinct colors. What is the maximum possible number of multichromatic colorings a positive integer can have if it is not the power of any prime?

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**9** How many ways can you fill a $3\times 3$ square grid with nonnegative integers such that no *nonzero* integer appears more than once in the same row or column and the sum of the numbers in every row and column equals 7?

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**10** Fred the Four-Dimensional Fluffy Sheep is walking in 4-dimensional space. He starts at the origin. Each minute, he walks from his current position $(a_1,a_2,a_3,a_4)$ to some position $(x_1,x_2,x_3,x_4)$ with integer coordinates satisfying

$$(x_1-a_1)^2+(x_2-a_2)^2+(x_3-a_3)^2+(x_4-a_4)^2=4 \quad \text{and} \quad |(x_1+x_2+x_3+x_4)-(a_1+a_2+a_3+a_4)|=2.$$

In how many ways can Fred reach $(10, 10, 10, 10)$ after exactly 40 minutes, if he is allowed to pass through this point during his walk?

– Geometry

**1** Let $d$ be a real number such that every non-degenerate quadrilateral has at least two interior angles with measure less than $d$ degrees. What is the minimum possible value for $d$?

**2** In rectangle $ABCD$, points $E$ and $F$ lie on sides $AB$ and $CD$ respectively such that both $AF$ and $CE$ are perpendicular to diagonal $BD$. Given that $BF$ and $DE$ separate $ABCD$ into three polygons with equal area, and that $EF = 1$, find the length of $BD$.

**3** Let $AB$ be a line segment with length 2, and $S$ be the set of points $P$ on the plane such that there exists point $X$ on segment $AB$ with $AX = 2PX$. Find the area of $S$.

**4** Convex hexagon $ABCDEF$ is drawn in the plane such that $ACDF$ and $ABDE$ are parallelograms with area 168. $AC$ and $BD$ intersect at $G$. Given that the area of $AGB$ is 10 more than the area of $CGB$, find the smallest possible area of hexagon $ABCDEF$.

**5** Isosceles triangle $ABC$ with $AB = AC$ is inscibed is a unit circle $\Omega$ with center $O$. Point $D$ is the reflection of $C$ across $AB$. Given that $DO = \sqrt{3}$, find the area of triangle $ABC$.

**6** Six unit disks $C_1$, $C_2$, $C_3$, $C_4$, $C_5$, $C_6$ are in the plane such that they don't intersect each other and $C_i$ is tangent to $C_{i+1}$ for $1 \leq i \leq 6$ (where $C_7 = C_1$). Let $C$ be the smallest circle that contains all six disks. Let $r$ be the smallest possible radius of $C$, and $R$ the largest possible radius. Find $R - r$.

**7** Let $ABC$ be a triangle with $AB = 13$, $BC = 14$, $CA = 15$. Let $H$ be the orthocenter of $ABC$. Find the radius of the circle with nonzero radius tangent to the circumcircles of $AHB$, $BHC$, $CHA$.

**8** In triangle $ABC$ with $AB < AC$, let $H$ be the orthocenter and $O$ be the circumcenter. Given that the midpoint of $OH$ lies on $BC$, $BC = 1$, and the perimeter of $ABC$ is 6, find the area of $ABC$.

**9** In a rectangular box $ABCDEFGH$ with edge lengths $AB = AD = 6$ and $AE = 49$, a plane slices through point $A$ and intersects edges $BF$, $FG$, $GH$, $HD$ at points $P$, $Q$, $R$, $S$ respectively. Given that $AP = AS$ and $PQ = QR = RS$, find the area of pentagon $APQRS$.

**10** In triangle $ABC$, $AB = 13$, $BC = 14$, $CA = 15$. Squares $ABB_1A_2$, $BCC_1B_2$, $CAA_1B_2$ are constructed outside the triangle. Squares $A_1A_2A_3A_4$, $B_1B_2B_3B_4$ are constructed outside the hexagon $A_1A_2B_1B_2C_1C_2$. Squares $A_3B_4B_5A_6$, $B_3C_4C_5B_6$, $C_3A_4A_5C_6$ are constructed outside the hexagon $A_4A_3B_4B_3C_4C_3$. Find the area of the hexagon $A_5A_6B_5B_6C_5C_6$.

 