Serbia National Math Olympiad 2015

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by mihajlon, IgorM, junioragd

–	Day 1
1	Consider circle inscribed quadriateral $ABCD$. Let M, N, P, Q be midpoints of sides DA, AB, BC, CD.Let E be the point of intersection of diagonals. Let $k1, k2$ be circles around EMN and EPQ . Let F be point of intersection of $k1$ and $k2$ different from E. Prove that EF is perpendicular to AC.
2	Let k be fixed positive integer . Let $Fk(n)$ be smallest positive integer bigger than kn such that $Fk(n) * n$ is a perfect square . Prove that if $Fk(n) = Fk(m)$ than $m = n$.
3	We have 2015 prisinoers.The king gives everyone a hat coloured in one of 5 colors.Everyone sees all hats expect his own.Now,the King orders them in a line(a prisioner can see all guys behind and in front of him).The king asks the prisinoers one by one does he know the color of his hat.If he answers **NO**,then he is killed.If he answers **YES**,then answers which color is his hat,if his answers is true,he goes to freedom,if not,he is killed.All the prisinors can hear did he answer **YES** or **NO**,but if he answered **YES**,they don't know what did he answered(he is killed in public).They can think of a strategy before the King comes,but after that they can't comunicate.What is the largest number of prisinors we can guarentee that can survive?
–	Day 2
4	For integer a, $a \neq 0$, $v_2(a)$ is greatest nonnegative integer k such that $2^k \vert a$. For given $n \in \mathbb{N}$ determine highest possible cardinality of subset A of set $\{1, 2, 3, ..., 2^n\}$ with following property: For all $x, y \in A$, $x \neq y$, number $v_2(x - y)$ is even.
5	Let x, y, z be nonnegative positive integers. Prove $\frac{x-y}{xy+2y+1} + \frac{y-z}{zy+2z+1} + \frac{z-x}{xz+2x+1} \geq 0$
6	In nonnegative set of integers solve the equation: $$(2^{2015} + 1)^x + 2^{2015} = 2^y + 1$$