**11th RMM 2019**

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| – | Day 1 |
| 1 | Amy and Bob play the game. At the beginning, Amy writes down a positive integer on the board. Then the players take moves in turn, Bob moves first. On any move of his, Bob replaces the number $n$ on the blackboard with a number of the form $n-a^2$, where $a$ is a positive integer. On any move of hers, Amy replaces the number $n$ on the blackboard with a number of the form $n^k$, where $k$ is a positive integer. Bob wins if the number on the board becomes zero.<br>Can Amy prevent Bobs win?<br>*Maxim Didin, Russia* |
| 2 | Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. Let $E$ be the midpoint of $AC$. Denote by $\omega$ and $\Omega$ the circumcircles of the triangles $ABE$ and $CDE$, respectively. Let $P$ be the crossing point of the tangent to $\omega$ at $A$ with the tangent to $\Omega$ at $D$. Prove that $PE$ is tangent to $\Omega$.<br>*Jakob Jurij Snoj, Slovenia* |
| 3 | Given any positive real number $\varepsilon$, prove that, for all but finitely many positive integers $v$, any graph on $v$ vertices with at least $(1+\varepsilon)v$ edges has two distinct simple cycles of equal lengths. (Recall that the notion of a simple cycle does not allow repetition of vertices in a cycle.)<br>*Fedor Petrov, Russia* |
| – | Day 2 |
| 4 | Prove that for every positive integer $n$ there exists a (not necessarily convex) polygon with no three collinear vertices, which admits exactly $n$ diffferent triangulations.<br>(A *triangulation* is a dissection of the polygon into triangles by interior diagonals which have no common interior points with each other nor with the sides of the polygon) |
| 5 | Determine all functions $f:\mathbb{R}\to\mathbb{R}$ satisfying $$f(x+yf(x))+f(xy)=f(x)+f(2019y),$$ for all real numbers $x$ and $y$. |
| 6 | Find all pairs of integers $(c, d)$, both greater than 1, such that the following holds: |

For any monic polynomial $Q$ of degree $d$ with integer coefficients and for any prime $p > c(2c+1)$, there exists a set $S$ of at most $(\frac{2c-1}{2c+1})p$ integers, such that

$$\bigcup_{s\in S}\{s,\ Q(s),\ Q(Q(s)),\ Q(Q(Q(s))),\ \dots\}$$

contains a complete residue system modulo $p$ (i.e., intersects with every residue class modulo $p$).


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