IX Iberoamerican Interuniversitary Mathematics Competition - Quito, Ecuador

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Problem 1 Determine all the complex numbers $w = a + bi$ with $a, b \in \mathbb{R}$, such that there exists a polinomial $p(z)$ whose coefficients are real and positive such that $p(w) = 0$.

Problem 2 Let $f : \mathbb{R} \to \mathbb{R}$ a derivable function such that $f(0) = 0$ and $|f'(x)| \leq |f(x) \cdot log|f(x)||$ for every $x \in \mathbb{R}$ such that $0 < |f(x)| < 1/2$. Prove that $f(x) = 0$ for every $x \in \mathbb{R}$.

Problem 3 Let G be a finite abelian group and $f : \mathbb{Z}^+ \to G$ a completely multiplicative function (i.e. $f(mn) = f(m)f(n)$ for any positive integers m, n). Prove that there are infinitely many positive integers k such that $f(k) = f(k + 1)$.

Problem 4 Let m, n be positive integers and $a_1, \dots, a_m, b_1, \dots, b_n$ positive real numbers such that for every positive integer k we have that

$$(a_1^k + \cdots + a_m^k) - (b_1^k + \cdots + b_n^k) \leq CkN,$$

for some fix C and N. Show that there exists $l \leq m, n$ and permutations σ of $\{1, \dots, m\}$ and τ of $\{1, \dots, n\}$, such that

1. $a\sigma(i) = b\tau(i)$ for $1 \leq i \leq l$,

2. $a\sigma(i), b\tau(i) \leq 1$ for $i > l$.

Problem 5 Let $\mathcal{S}$ be a set of integers. Given a real positive r, we say that $\mathcal{S}$ is a r-discerning, if for any pair $m, n > 1$ of distinct integers such that $\left|\frac{m-n}{m+n}\right| < r$, there exists $a \in \mathcal{S}$ and $k \geq 1$ such that a^k divides m but not n, or a^k divides n but not m

1. Show that for every $r > 0$ every r-discerning set contains an infinite number of primes.

2. For every $r > 0$ determine the maximal possible cardinality of $\mathcal{P} \backslash \mathcal{S}$ where $\mathcal{P}$ is the set of primes and $\mathcal{S} \subseteq \mathcal{P}$ is a r-discerning set.

Problem 6 Let G be a simple, connected and finite grafo. A hunter and an invisible rabbit play in the graph G.
The rabbit is initially in a vertex w_0. In the k-th turn (for $k \geq 0$) the hunter picks freely a vertex v_k. If $v_k = w_k$, the rabbit is capture and the game ends. If not, the rabbit moves invisibly by an edge of w_k to w_{k+1} (w_k and w_{k+1} are adjacent and therefore distinct) and the game continues. The hunter knows these rules and the graph G. After the $k-$th turn he knows that $w_k \neq v_k$, but he gets no more information.
Characterize the graphs G such that the hunter has an strategy that guaranties that he can

capture the rabbit in at most N turns for some positive integer N. Here N must depend only on G and the strategy should work independently of the initial position and trajectory of the rabbit.