Moldova Team Selection Test 2019

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– Day 1

1 Let S be the set of all natural numbers with the property: the sum of the biggest three divisors of number n, different from n, is bigger than n. Determine the largest natural number k, which divides any number from S.
(A natural number is a positive integer)

2 Prove that $E_n=\frac{\arccos\frac{n-1}{n}}{\text{arccot}\sqrt{2n-1}}$ is a natural number for any natural number n.
(A natural number is a positive integer)

3 On the table there are written numbers $673, 674, \cdots, 2018, 2019$. Nibab chooses arbitrarily three numbers a, b and c, erases them and writes the number $\frac{\min(a,b,c)}{3}$, then he continues in an analogous way. After Nibab performed this operation 673 times, on the table remained a single number k. Prove that $k \in (0,1)$.

4 Quadrilateral $ABCD$ is inscribed in circle Γ with center O. Point I is the incenter of triangle ABC, and point J is the incenter of the triangle ABD. Line IJ intersects segments AD, AC, BD, BC at points P, M, N and, respectively Q. The perpendicular from M to line AC intersects the perpendicular from N to line BD at point X. The perpendicular from P to line AD intersects the perpendicular from Q to line BC at point Y. Prove that X, O, Y are colinear.

– Day 2

5 Point H is the orthocenter of the scalene triangle ABC. A line, which passes through point H, intersect the sides AB and AC at points D and E, respectively, such that $AD = AE$. Let M be the midpoint of side BC. Line MH intersects the circumscribed circle of triangle ABC at point K, which is on the smaller arc AB. Prove that Nibab can draw a circle through A, D, E and K.

6 Let $a, b, c \geq 0$ such that $a+b+c=1$ and $s \geq 5$.
Prove that $s(a^2+b^2+c^2) \leq 3(s-3)(a^3+b^3+c^3)+1$

7 Let $P(X)=a_{2n+1}X^{2n+1}+a_{2n}X^{2n}+...+a_1X+a_0$ be a polynomial with all positive coefficients. Prove that there exists a permutation $(b_{2n+1}, b_{2n}, ..., b_1, b_0)$ of numbers $(a_{2n+1}, a_{2n}, ..., a_1, a_0)$ such that the polynomial $Q(X)=b_{2n+1}X^{2n+1}+b_{2n}X^{2n}+...+b_1X+b_0$ has exactly one real root.

8 For any positive integer k denote by $S(k)$ the number of solutions $(x,y) \in \mathbb{Z}_+ \times \mathbb{Z}_+$ of the system

$$\begin{cases} \left\lceil \frac{x \cdot d}{y} \right\rceil \cdot \frac{x}{d} = \left\lceil \left(\sqrt{y}+1\right)^2 \right\rceil \\ \mid x-y \mid = k, \end{cases}$$

where d is the greatest common divisor of positive integers x and y. Determine $S(k)$ as a function of k. (Here $\lceil z \rceil$ denotes the smalles integer number which is bigger or equal than z.)

– Day 3

9 Find all polynomials $P(X)$ with real coefficients such that if real numbers x, y and z satisfy $x+y+z=0,$ then the points $\left(x, P(x)\right), \left(y, P(y)\right), \left(z, P(z)\right)$ are all colinear.

10 The circle Ω with center O is circumscribed to acute triangle ABC. Let P be a point on the circumscribed circle of OBC, such that P is inside ABC and is different from B and C. Bisectors of angles BPA and CPA intersect the sides AB and AC in points E and F. Prove that the incenters of triangles PEF, PCA and PBA are collinear.

11 Let $n \geq 2,$ be a positive integer. Numbers $\{1, 2, 3, ..., n\}$ are written in a row in an arbitrary order. Determine the smalles positive integer k with the property: everytime it is possible to delete k numbers from those written on the table, such that the remained numbers are either in an increasing or decreasing order.

12 Let $p \geq 5$ be a prime number. Prove that there exist positive integers m and n with $m+n \leq \frac{p+1}{2}$ for which p divides $2^n \cdot 3^m - 1.$