X Iberoamerican Interuniversitary Mathematics Competition - Colombia

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by Ozc

Problem 1 Show that there exists a 2×2 matrix of order 6 with rational entries, such that the sum of its entries is 2018.
Note: The order of a matrix (if it exists) is the smallest positive integer n such that $A^n = I$, where I is the identity matrix.

Problem 2 Let $p(x)$ and $q(x)$ non constant real polynomials of degree at most n ($n > 1$). Show that there exists a non zero polynomial $F(x, y)$ in two variables with real coefficients of degree at most $2n - 2$, such that $F(p(t), q(t)) = 0$ for every $t \in \mathbb{R}$.

Problem 3 Let m be an integer and $\mathbb{Z}_m$ the set of integer modulo m. An equivalence relation is defined in $\mathbb{Z}_m$ given by, $x \sim y$ if there exists a natural t such that $y \equiv 2^t x \pmod{m}$. Find al values of m such that the number of equivalent classes is even.

Problem 4 Let $\alpha < 0 < \beta$ and consider the polynomial $f(x) = x(x - \alpha)(x - \beta)$. Let S be the set of real numbers s such that $f(x) - s$ has three different real roots. For $s \in S$, let $p(x)$ the product of the smallest and largest root of $f(x) - s$. Determine the smallest possible value that $p(s)$ for $s \in S$.

Problem 5 Consider the transformation

$$T(x, y, z) = (\sin y + \sin z - \sin x, \sin z + \sin x - \sin y, \sin x + \sin y - \sin z).$$

Determine all the points $(x, y, z) \in [0, 1]^3$ such that $T^n(x, y, z) \in [0, 1]^3$, for every $n \geq 1$.

Problem 6 Let $\{x_n\}$ be a sequence of real numbers in the interval $[0, 1)$. Prove that there exists a sequence $1 < n_1 < n_2 < n_3 < \cdots$ of positive integers such that the following limit exists

$$\lim_{i,j \to \infty} x_{n_i + n_j}.$$

That is, there exists a real number L such that for every $\epsilon > 0$, there exists a positive integer N such that if $i, j > N$, then $|x_{n_i + n_j} - L| < \epsilon$.
