Moldova Team Selection Test 2016

www.artofproblemsolving.com/community/c845142

by Snakes, Ghd, Peter, augustin_p, XxProblemDestroyer1337xX, rkm0959

–	Day 1
1	If $x_1, x_2, ..., x_n > 0$ and $x_1^2 + x_2^2 + ... + x_n^2 = \frac{1}{n}$,prove that $\sum x_i + \sum \frac{1}{x_i \cdot x_{i+1}} \geq n^3 + 1$.
2	Let p be a prime number of the form $4k+1$. Show that $$\sum_{i=1}^{p-1}\left(\left\lfloor\frac{2i^2}{p}\right\rfloor-2\left\lfloor\frac{i^2}{p}\right\rfloor\right)=\frac{p-1}{2}.$$
3	Let ABC be a triangle with $\angle C = 90$. The tangent points of the inscribed circle with the sides BC, CA and AB are M, N and P. Points M_1, N_1, P_1 are symmetric to points M, N, P with respect to midpoints of sides BC, CA and AB. Find the smallest value of $\frac{AO_1 + BO_1}{AB}$, where O_1 is the circumcenter of triangle $M_1N_1P_1$.
4	Show that for every prime number p and every positive integer $n \geq 2$ there exists a positive integer k such that the decimal representation of p^k contains n consecutive equal digits.
–	Day 2
5	The sequence of polynomials $(P_n(X))_{n \in Z_{>0}}$ is defined as follows: $P_1(X) = 2X$ $P_2(X) = 2(X^2 + 1)$ $P_{n+2}(X) = 2X \cdot P_{n+1}(X) - (X^2 - 1)P_n(X)$, for all positive integers n. Find all n for which $X^2 + 1 \mid P_n(X)$
6	Let $n \in \mathbb{Z}_{>0}$. The set S contains all positive integers written in decimal form that simultaneously satisfy the following conditions: - each element of S has exactly n digits; - each element of S is divisible by 3; - each element of S has all its digits from the set $\{3, 5, 7, 9\}$ Find $\mid S \mid$
7	Let Ω and O be the circumcircle of acute triangle ABC and its center, respectively. $M \neq O$ is an arbitrary point in the interior of ABC such that AM, BM, and CM intersect Ω at A_1, B_1, and C_1, respectiuvely. Let A_2, B_2, and C_2 be the circumcenters of MBC, MCA, and MAB, respectively. It is to be proven that A_1A_2, B_1B_2, C_1C2 concur.

© 2023 AoPS Incorporated

8 Let us have n ($n > 3$) balls with different rays. On each ball it is written an integer number. Determine the greatest natural number d such that for any numbers written on the balls, we can always find at least 4 different ways to choose some balls with the sum of the numbers written on them divisible by d.

– Day 3

9 Let $\alpha \in \left(0, \frac{\pi}{2}\right)$.Find the minimum value of the expression

$$P = (1+\cos\alpha)\left(1+\frac{1}{\sin\alpha}\right) + (1+\sin\alpha)\left(1+\frac{1}{\cos\alpha}\right).$$

10 Let $A_1A_2\cdots A_{14}$ be a regular $14-$gon. Prove that $A_1A_3 \cap A_5A_{11} \cap A_6A_9 \neq \emptyset$.

11 Let $ABCD$ be a cyclic quadrilateral. Circle with diameter AB intersects CA, CB, DA, and DB in E, F, G, and H, respectively (all different from A and B). The lines EF and GH intersect in I. Prove that the bisector of $\angle GIF$ and the line CD are perpendicular.

12 There are 2015 distinct circles in a plane, with radius 1.
Prove that you can select 27 circles, which form a set C, which satisfy the following.

For two arbitrary circles in C, they intersect with each other or
For two arbitrary circles in C, they don't intersect with each other.

© 2023 AoPS Incorporated

Art of Problem Solving is an ACS WASC Accredited School.