AIME Problems 2019

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–	I
–	March 13th, 2019
1	Consider the integer $$N = 9 + 99 + 999 + 9999 + \cdots + \underbrace{99\ldots 99}_{\text{321 digits}}.$$ Find the sum of the digits of N.
2	Jenn randomly chooses a number J from $1, 2, 3, \ldots, 19, 20$. Bela then randomly chooses a number B from $1, 2, 3, \ldots, 19, 20$ distinct from J. The value of $B - J$ is at least 2 with a probability that can be expressed in the form $\frac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.
3	In $\triangle PQR$, $PR = 15$, $QR = 20$, and $PQ = 25$. Points A and B lie on $\overline{PQ}$, points C and D lie on $\overline{QR}$, and points E and F lie on $\overline{PR}$, with $PA = QB = QC = RD = RE = PF = 5$. Find the area of hexagon $ABCDEF$.
4	A soccer team has 22 available players. A fixed set of 11 players starts the game, while the other 11 are available as substitutes. During the game, the coach may make as many as 3 substitutions, where any one of the 11 players in the game is replaced by one of the substitutes. No player removed from the game may reenter the game, although a substitute entering the game may be replaced later. No two substitutions can happen at the same time. The players involved and the order of the substitutions matter. Let n be the number of ways the coach can make substitutions during the game (including the possibility of making no substitutions). Find the remainder when n is divided by 1000.
5	A moving particle starts at the point $(4, 4)$ and moves until it hits one of the coordinate axes for the first time. When the particle is at the point (a, b), it moves at random to one of the points $(a - 1, b)$, $(a, b - 1)$, or $(a - 1, b - 1)$, each with probability $\frac{1}{3}$, independently of its previous moves. The probability that it will hit the coordinate axes at $(0, 0)$ is $\frac{m}{3^n}$, where m and n are positive integers, and m is not divisible by 3. Find $m + n$.
6	In convex quadrilateral $KLMN$ side $\overline{MN}$ is perpendicular to diagonal $\overline{KM}$, side $\overline{KL}$ is perpendicular to diagonal $\overline{LN}$, $MN = 65$, and $KL = 28$. The line through L perpendicular to side $\overline{KN}$ intersects diagonal $\overline{KM}$ at O with $KO = 8$. Find MO.

7 There are positive integers x and y that satisfy the system of equations

$$\log_{10} x + 2\log_{10}(\gcd(x,y)) = 60$$
$$\log_{10} y + 2\log_{10}(\text{lcm}(x,y)) = 570.$$

Let m be the number of (not necessarily distinct) prime factors in the prime factorization of x, and let n be the number of (not necessarily distinct) prime factors in the prime factorization of y. Find $3m+2n$.

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8 Let x be a real number such that $\sin^{10} x + \cos^{10} x = \frac{11}{36}$. Then $\sin^{12} x + \cos^{12} x = \frac{m}{n}$ where m and n are relatively prime positive integers. Find $m+n$.

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9 Let $\tau(n)$ denote the number of positive integer divisors of n. Find the sum of the six least positive integers n that are solutions to $\tau(n) + \tau(n+1) = 7$.

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10 For distinct complex numbers $z_1, z_2, \dots, z_{673}$, the polynomial

$$(x-z_1)^3(x-z_2)^3\cdots(x-z_{673})^3$$

can be expressed as $x^{2019} + 20x^{2018} + 19x^{2017} + g(x)$, where $g(x)$ is a polynomial with complex coefficients and with degree at most 2016. The value of

$$\left|\sum_{1\le j<k\le 673} z_j z_k\right|$$

can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m+n$.

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11 In $\triangle ABC$, the sides have integers lengths and $AB = AC$. Circle ω has its center at the incenter of $\triangle ABC$. An *excircle* of $\triangle ABC$ is a circle in the exterior of $\triangle ABC$ that is tangent to one side of the triangle and tangent to the extensions of the other two sides. Suppose that the excircle tangent to $\overline{BC}$ is internally tangent to ω, and the other two excircles are both externally tangent to ω. Find the minimum possible value of the perimeter of $\triangle ABC$.

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12 Given $f(z) = z^2 - 19z$, there are complex numbers z with the property that z, $f(z)$, and $f(f(z))$ are the vertices of a right triangle in the complex plane with a right angle at $f(z)$. There are positive integers m and n such that one such value of z is $m + \sqrt{n} + 11i$. Find $m+n$.

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13 Triangle ABC has side lengths $AB = 4$, $BC = 5$, and $CA = 6$. Points D and E are on ray AB with $AB < AD < AE$. The point $F \neq C$ is a point of intersection of the circumcircles of $\triangle ACD$ and $\triangle EBC$ satisfying $DF = 2$ and $EF = 7$. Then BE can be expressed as $\frac{a+b\sqrt{c}}{d}$, where a, b, c, and d are positive integers such that a and d are relatively prime, and c is not divisible by the square of any prime. Find $a+b+c+d$.

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14 Find the least odd prime factor of $2019^8 + 1$.

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15 Let $\overline{AB}$ be a chord of a circle ω, and let P be a point on the chord $\overline{AB}$. Circle ω_1 passes through A and P and is internally tangent to ω. Circle ω_2 passes through B and P and is internally tangent to ω. Circles ω_1 and ω_2 intersect at points P and Q. Line PQ intersects ω at X and Y. Assume that $AP = 5$, $PB = 3$, $XY = 11$, and $PQ^2 = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

– II

– March 21st, 2019

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1 Points $C \neq D$ lie on the same side of line AB so that $\triangle ABC$ and $\triangle BAD$ are congruent with $AB = 9$, $BC = AD = 10$, and $CA = DB = 17$. The intersection of these two triangular regions has area $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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2 Lily pads $1, 2, 3, \ldots$ lie in a row on a pond. A frog makes a sequence of jumps starting on pad 1. From any pad k the frog jumps to either pad $k+1$ or pad $k+2$ chosen randomly and independently with probability $\frac{1}{2}$. The probability that the frog visits pad 7 is $\frac{p}{q}$, where p and q are relatively prime positive integers. Find $p + q$.

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3 Find the number of 7-tuples of positive integers (a, b, c, d, e, f, g) that satisfy the following systems of equations:

$$abc = 70,$$
$$cde = 71,$$
$$efg = 72.$$

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4 A standard six-sided fair die is rolled four times. The probability that the product of all four numbers rolled is a perfect square is $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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5 Four ambassadors and one advisor for each of them are to be seated at a round table with 12 chairs numbered in order from 1 to 12. Each ambassador must sit in an even-numbered chair. Each advisor must sit in a chair adjacent to his or her ambassador. There are N ways for the 8 people to be seated at the table under these conditions. Find the remainder when N is divided by 1000.

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6 In a Martian civilization, all logarithms whose bases are not specified are assumed to be base b, for some fixed $b \geq 2$. A Martian student writes down

$$3\log(\sqrt{x}\log x) = 56$$
$$\log_{\log(x)}(x) = 54$$

and finds that this system of equations has a single real number solution $x > 1$. Find b.

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7 Triangle ABC has side lengths $AB = 120$, $BC = 220$, and $AC = 180$. Lines ℓ_A, ℓ_B, and ℓ_C are drawn parallel to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively, such that the intersection of ℓ_A, ℓ_B, and ℓ_C with the interior of $\triangle ABC$ are segments of length 55, 45, and 15, respectively. Find the perimeter of the triangle whose sides lie on ℓ_A, ℓ_B, and ℓ_C.

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8 The polynomial $f(z) = az^{2018} + bz^{2017} + cz^{2016}$ has real coefficients not exceeding 2019, and $f(\frac{1+\sqrt{3}i}{2}) = 2015 + 2019\sqrt{3}i$. Find the remainder when $f(1)$ is divided by 1000.

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9 Call a positive integer n k-*pretty* if n has exactly k positive divisors and n is divisible by k. For example, 18 is 6-*pretty*. Let S be the sum of positive integers less than 2019 that are 20-*pretty*. Find $\frac{S}{20}$.

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10 There is a unique angle θ between 0° and 90° such that for nonnegative integers n, the value of $\tan(2^n\theta)$ is positive when n is a multiple of 3, and negative otherwise. The degree measure of θ is $\frac{p}{q}$, where p and q are relatively prime integers. Find $p + q$.

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11 Triangle ABC has side lengths $AB = 7, BC = 8,$ and $CA = 9$. Circle ω_1 passes through B and is tangent to line AC at A. Circle ω_2 passes through C and is tangent to line AB at A. Let K be the intersection of circles ω_1 and ω_2 not equal to A. Then $AK = \frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

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12 For $n \geq 1$ call a finite sequence $(a_1, a_2 \ldots a_n)$ of positive integers *progressive* if $a_i < a_{i+1}$ and a_i divides a_{i+1} for all $1 \leq i \leq n - 1$. Find the number of progressive sequences such that the sum of the terms in the sequence is equal to 360.

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13 Regular octagon $A_1A_2A_3A_4A_5A_6A_7A_8$ is inscribed in a circle of area 1. Point P lies inside the circle so that the region bounded by $\overline{PA_1}$, $\overline{PA_2}$, and the minor arc $\widehat{A_1A_2}$ of the circle has area $\frac{1}{7}$, while the region bounded by $\overline{PA_3}$, $\overline{PA_4}$, and the minor arc $\widehat{A_3A_4}$ of the circle has area $\frac{1}{9}$. There is a positive integer n such that the area of the region bounded by $\overline{PA_6}$, $\overline{PA_7}$, and the minor arc $\widehat{A_6A_7}$ is equal to $\frac{1}{8} - \frac{\sqrt{2}}{n}$. Find n.

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14 Find the sum of all positive integers n such that, given an unlimited supply of stamps of denominations 5, n, and $n + 1$ cents, 91 cents is the greatest postage that cannot be formed.

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15 In acute triangle ABC points P and Q are the feet of the perpendiculars from C to $\overline{AB}$ and from B to $\overline{AC}$, respectively. Line PQ intersects the circumcircle of $\triangle ABC$ in two distinct points, X and Y. Suppose $XP = 10$, $PQ = 25$, and $QY = 15$. The value of $AB \cdot AC$ can be written in the form $m\sqrt{n}$ where m and n are positive integers, and n is not divisible by the square of any prime. Find $m + n$.