China Team Selection Test 2019

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– **Test 1** Day 1

1 $ABCDE$ is a cyclic pentagon, with circumcentre O. $AB=AE=CD$. I midpoint of BC. J midpoint of DE. F is the orthocentre of $\triangle ABE$, and G the centroid of $\triangle AIJ$.CE intersects BD at H, OG intersects FH at M. Show that $AM\perp CD$.

2 Fix a positive integer $n\ge 3$. Does there exist infinitely many sets S of positive integers $\{a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n\}$, such that $\gcd(a_1,a_2,\ldots,a_n,b_1,b_2,\ldots,b_n)=1$, $\{a_i\}_{i=1}^n$, $\{b_i\}_{i=1}^n$ are arithmetic progressions, and $\prod_{i=1}^n a_i=\prod_{i=1}^n b_i$?

3 Find all positive integer n, such that there exists n points $P_1,\ldots,P_n$ on the unit circle , satisfying the condition that for any point M on the unit circle, $\sum_{i=1}^n MP_i^k$ is a fixed value for

a) $k=2018$

b) $k=2019$.

– **Test 1** Day 2

4 Call a sequence of positive integers $\{a_n\}$ good if for any distinct positive integers m,n, one has

$$\gcd(m,n)\mid a_m^2+a_n^2 \text{ and } \gcd(a_m,a_n)\mid m^2+n^2.$$

Call a positive integer a to be k-good if there exists a good sequence such that $a_k=a$. Does there exists a k such that there are exactly 2019 k-good positive integers?

5 Determine all functions $f:\mathbb{Q}\to\mathbb{Q}$ such that

$$f(2xy+\frac{1}{2})+f(x-y)=4f(x)f(y)+\frac{1}{2}$$

for all $x,y\in\mathbb{Q}$.

6 Let k be a positive real. A and B play the following game: at the start, there are 80 zeroes arrange around a circle. Each turn, A increases some of these 80 numbers, such that the total sum added is 1. Next, B selects ten consecutive numbers with the largest sum, and reduces them all to 0. A then wins the game if he/she can ensure that at least one of the number is $\ge k$ at some finite point of time.

Determine all k such that A can always win the game.

–	**Test 2** Day 1
1	AB and AC are tangents to a circle ω with center O at B, C respectively. Point P is a variable point on minor arc BC. The tangent at P to ω meets AB, AC at D, E respectively. AO meets BP, CP at U, V respectively. The line through P perpendicular to AB intersects DV at M, and the line through P perpendicular to AC intersects EU at N. Prove that as P varies, MN passes through a fixed point.
2	Let S be the set of 10-tuples of non-negative integers that have sum 2019. For any tuple in S, if one of the numbers in the tuple is ≥ 9, then we can subtract 9 from it, and add 1 to the remaining numbers in the tuple. Call thus one operation. If for $A, B \in S$ we can get from A to B in finitely many operations, then denote $A \rightarrow B$. (1) Find the smallest integer k, such that if the minimum number in $A, B \in S$ respectively are both $\geq k$, then $A \rightarrow B$ implies $B \rightarrow A$. (2) For the k obtained in (1), how many tuples can we pick from S, such that any two of these tuples A, B that are distinct, $A \not\rightarrow B$.
3	Let n be a given even number, $a_1, a_2, \cdots, a_n$ be non-negative real numbers such that $a_1 + a_2 + \cdots + a_n = 1$. Find the maximum possible value of $\sum_{1\leq i<j\leq n} \min\{(i-j)^2, (n+i-j)^2\} a_i a_j$.
–	**Test 2** Day 2
4	Does there exist a finite set A of positive integers of at least two elements and an infinite set B of positive integers, such that any two distinct elements in $A+B$ are coprime, and for any coprime positive integers m, n, there exists an element x in $A+B$ satisfying $x \equiv n \pmod m$? Here $A+B = \{a+b \| a \in A, b \in B\}$.
5	Let M be the midpoint of BC of triangle ABC. The circle with diameter BC, ω, meets AB, AC at D, E respectively. P lies inside $\triangle ABC$ such that $\angle PBA = \angle PAC, \angle PCA = \angle PAB$, and $2PM \cdot DE = BC^2$. Point X lies outside ω such that $XM \parallel AP$, and $\frac{XB}{XC} = \frac{AB}{AC}$. Prove that $\angle BXC + \angle BAC = 90^\circ$.
6	Given coprime positive integers $p, q > 1$, call all positive integers that cannot be written as $px + qy$(where x, y are non-negative integers) *bad*, and define $S(p, q)$ to be the sum of all bad numbers raised to the power of 2019. Prove that there exists a positive integer n, such that for any p, q as described, $(p-1)(q-1)$ divides $nS(p, q)$.
–	**Test 3** Day 1

1 Given complex numbers x, y, z, with $|x|^2+|y|^2+|z|^2=1$. Prove that:
$$|x^3+y^3+z^3-3xyz|\le 1$$

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2 Let S be a set of positive integers, such that $n\in S$ if and only if
$$\sum_{d|n,d<n,d\in S} d\le n$$
Find all positive integers $n=2^k\cdot p$ where k is a non-negative integer and p is an odd prime, such that
$$\sum_{d|n,d<n,d\in S} d=n$$

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3 Does there exist a bijection $f:\mathbb{N}^+\to\mathbb{N}^+$, such that there exist a positive integer k, and it's possible to have each positive integer colored by one of k chosen colors, such that for any $x\neq y$, $f(x)+y$ and $f(y)+x$ are not the same color?

– **Test 3** Day 2

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4 Find all functions $f:\mathbb{R}^2\to\mathbb{R}$, such that
1) $f(0,x)$ is non-decreasing ;
2) for any $x,y\in\mathbb{R}$, $f(x,y)=f(y,x)$;
3) for any $x,y,z\in\mathbb{R}$, $(f(x,y)-f(y,z))(f(y,z)-f(z,x))(f(z,x)-f(x,y))=0$;
4) for any $x,y,a\in\mathbb{R}$, $f(x+a,y+a)=f(x,y)+a$.

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5 In ΔABC, $AD\perp BC$ at D. E,F lie on line AB, such that $BD=BE=BF$. Let I,J be the incenter and A-excenter. Prove that there exist two points P,Q on the circumcircle of ΔABC , such that $PB=QC$, and $\Delta PEI\sim\Delta QFJ$.

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6 Given positive integers $d\ge 3$, $r>2$ and l, with $2d\le l<rd$. Every vertice of the graph $G(V,E)$ is assigned to a positive integer in $\{1,2,\cdots,l\}$, such that for any two consecutive vertices in the graph, the integers they are assigned to, respectively, have difference no less than d, and no more than $l-d$.
A proper coloring of the graph is a coloring of the vertices, such that any two consecutive vertices are not the same color. It's given that there exist a proper subset A of V, such that for G's any proper coloring with $r-1$ colors, and for an arbitrary color C, either all numbers in color C appear in A, or none of the numbers in color C appear in A.
Show that G has a proper coloring within $r-1$ colors.

– **Test 4** Day 1

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1 Cyclic quadrilateral $ABCD$ has circumcircle (O). Points M and N are the midpoints of BC and CD, and E and F lie on AB and AD respectively such that EF passes through O and $EO = OF$. Let EN meet FM at P. Denote S as the circumcenter of $\triangle PEF$. Line PO intersects AD and BA at Q and R respectively. Suppose $OSPC$ is a parallelogram. Prove that $AQ = AR$.

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2 A graph $G(V,E)$ is triangle-free, but adding any edges to the graph will form a triangle. It's given that $|V| = 2019$, $|E| > 2018$, find the minimum of $|E|$.

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3 60 points lie on the plane, such that no three points are collinear. Prove that one can divide the points into 20 groups, with 3 points in each group, such that the triangles (20 in total) consist of three points in a group have a non-empty intersection.

– **Test 4** Day 2

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4 Prove that there exist a subset A of $\{1, 2, \cdots, 2^n\}$ with n elements, such that for any two different non-empty subset of A, the sum of elements of one subset doesn't divide another's.

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5 Find all integer n such that the following property holds: for any positive real numbers a, b, c, x, y, z, with $max(a, b, c, x, y, z) = a$, $a + b + c = x + y + z$ and $abc = xyz$, the inequality

$$a^n + b^n + c^n \geq x^n + y^n + z^n$$

holds.

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6 Given positive integer n, k such that $2 \leq n < 2^k$. Prove that there exist a subset A of $\{0, 1, \cdots, n\}$ such that for any $x \neq y \in A$, $\binom{y}{x}$ is even, and

$$|A| \geq \frac{\binom{k}{\lfloor \frac{k}{2} \rfloor}}{2^k} \cdot (n+1)$$
