Sharygin Geometry Olympiad 2019

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– First (Correspondence) Round

1 Let AA_1, CC_1 be the altitudes of ΔABC, and P be an arbitrary point of side BC. Point Q on the line AB is such that $QP = PC_1$, and point R on the line AC is such that $RP = CP$. Prove that QA_1RA is a cyclic quadrilateral.

2 The circle ω_1 passes through the center O of the circle ω_2 and meets it at points A and B. The circle ω_3 centered at A with radius AB meets ω_1 and ω_2 at points C and D (distinct from B). Prove that C, O, D are collinear.

3 The rectangle $ABCD$ lies inside a circle. The rays BA and DA meet this circle at points A_1 and A_2. Let A_0 be the midpoint of A_1A_2. Points B_0, C_0, D_0 are defined similarly. Prove that $A_0C_0 = B_0D_0$.

4 The side AB of ΔABC touches the corresponding excircle at point T. Let J be the center of the excircle inscribed into $\angle A$, and M be the midpoint of AJ. Prove that $MT = MC$.

5 Let A, B, C and D be four points in general position, and ω be a circle passing through B and C. A point P moves along ω. Let Q be the common point of circles $\odot(ABP)$ and $\odot(PCD)$ distinct from P. Find the locus of points Q.

6 Two quadrilaterals $ABCD$ and $A_1B_1C_1D_1$ are mutually symmetric with respect to the point P. It is known that A_1BCD, AB_1CD and ABC_1D are cyclic quadrilaterals. Prove that the quadrilateral $ABCD_1$ is also cyclic

7 Let AH_A, BH_B, CH_C be the altitudes of the acute-angled ΔABC. Let X be an arbitrary point of segment CH_C, and P be the common point of circles with diameters H_CX and BC, distinct from H_C. The lines CP and AH_A meet at point Q, and the lines XP and AB meet at point R. Prove that A, P, Q, R, H_B are concyclic.

8 The circle ω_1 passes through the vertex A of the parallelogram $ABCD$ and touches the rays CB, CD. The circle ω_2 touches the rays AB, AD and touches ω_1 externally at point T. Prove that T lies on the diagonal AC

9 Let A_M be the midpoint of side BC of an acute-angled ΔABC, and A_H be the foot of the altitude to this side. Points B_M, B_H, C_M, C_H are defined similarly. Prove that one of the ratios

$A_MA_H : A_HA, B_MB_H : B_HB, C_MC_H : C_HC$ is equal to the sum of two remaining ratios

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10 Let N be the midpoint of arc ABC of the circumcircle of ΔABC, and NP, NT be the tangents to the incircle of this triangle. The lines BP and BT meet the circumcircle for the second time at points P_1 and T_1 respectively. Prove that $PP_1 = TT_1$.

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11 Morteza marks six points in the plane. He then calculates and writes down the area of every triangle with vertices in these points (20 numbers). Is it possible that all of these numbers are integers, and that they add up to 2019?

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12 Let $A_1A_2A_3$ be an acute-angled triangle inscribed into a unit circle centered at O. The cevians from A_i passing through O meet the opposite sides at points B_i $(i = 1, 2, 3)$ respectively.

- Find the minimal possible length of the longest of three segments B_iO.
- Find the maximal possible length of the shortest of three segments B_iO.

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13 Let ABC be an acute-angled triangle with altitude $AT = h$. The line passing through its circumcenter O and incenter I meets the sides AB and AC at points F and N, respectively. It is known that $BFNC$ is a cyclic quadrilateral. Find the sum of the distances from the orthocenter of ABC to its vertices.

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14 Let the side AC of triangle ABC touch the incircle and the corresponding excircle at points K and L respectively. Let P be the projection of the incenter onto the perpendicular bisector of AC. It is known that the tangents to the circumcircle of triangle BKL at K and L meet on the circumcircle of ABC. Prove that the lines AB and BC touch the circumcircle of triangle PKL.

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15 The incircle ω of triangle ABC touches the sides BC, CA and AB at points D, E and F respectively. The perpendicular from E to DF meets BC at point X, and the perpendicular from F to DE meets BC at point Y. The segment AD meets ω for the second time at point Z. Prove that the circumcircle of the triangle XYZ touches ω.

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16 Let AH_1 and BH_2 be the altitudes of triangle ABC. Let the tangent to the circumcircle of ABC at A meet BC at point S_1, and the tangent at B meet AC at point S_2. Let T_1 and T_2 be the midpoints of AS_1 and BS_2 respectively. Prove that T_1T_2, AB and H_1H_2 concur.

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17 Three circles ω_1, ω_2, ω_3 are given. Let A_0 and A_1 be the common points of ω_1 and ω_2, B_0 and B_1 be the common points of ω_2 and ω_3, C_0 and C_1 be the common points of ω_3 and ω_1. Let $O_{i,j,k}$ be the circumcenter of triangle $A_iB_jC_k$. Prove that the four lines of the form $O_{ijk}O_{1-i,1-j,1-k}$ are concurrent or parallel.

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18 A quadrilateral $ABCD$ without parallel sidelines is circumscribed around a circle centered at I. Let K, L, M and N be the midpoints of AB, BC, CD and DA respectively. It is known that $AB \cdot CD = 4IK \cdot IM$. Prove that $BC \cdot AD = 4IL \cdot IN$.

19 Let AL_a, BL_b, CL_c be the bisecors of triangle ABC. The tangents to the circumcircle of ABC at B and C meet at point K_a, points K_b, K_c are defined similarly. Prove that the lines K_aL_a, K_bL_b and K_cL_c concur.

20 Let O be the circumcenter of triangle ABC, H be its orthocenter, and M be the midpoint of AB. The line MH meets the line passing through O and parallel to AB at point K lying on the circumcircle of ABC. Let P be the projection of K onto AC. Prove that $PH \parallel BC$.

21 An ellipse Γ and its chord AB are given. Find the locus of orthocenters of triangles ABC inscribed into Γ.

22 Let AA_0 be the altitude of the isosceles triangle ABC $(AB = AC)$. A circle γ centered at the midpoint of AA_0 touches AB and AC. Let X be an arbitrary point of line BC. Prove that the tangents from X to γ cut congruent segments on lines AB and AC

23 In the plane, let a, b be two closed broken lines (possibly self-intersecting), and K, L, M, N be four points. The vertices of a, b and the points K L, M, N are in general position (i.e. no three of these points are collinear, and no three segments between them concur at an interior point). Each of segments KL and MN meets a at an even number of points, and each of segments LM and NK meets a at an odd number of points. Conversely, each of segments KL and MN meets b at an odd number of points, and each of segments LM and NK meets b at an even number of points. Prove that a and b intersect.

24 Two unit cubes have a common center. Is it always possible to number the vertices of each cube from 1 to 8 so that the distance between each pair of identically numbered vertices would be at most $4/5$? What about at most $13/16$?

– **Final Round**

– Grade 8

1 A trapezoid with bases AB and CD is inscribed into a circle centered at O. Let AP and AQ be the tangents from A to the circumcircle of triangle CDO. Prove that the circumcircle of triangle APQ passes through the midpoint of AB.

2 A point M inside triangle ABC is such that $AM = AB/2$ and $CM = BC/2$. Points C_0 and A_0 lying on AB and CB respectively are such that $BC_0 : AC_0 = BA_0 : CA_0 = 3$. Prove that the distances from M to C_0 and A_0 are equal.

3 Construct a regular triangle using a plywood square. (*You can draw a line through pairs of points lying on the distance less than the side of the square, construct a perpendicular from a point to the* *line the distance between them does not exceed the side of the square, and measure segments on the constructed lines equal to the side or to the diagonal of the square*)

4 Let O, H be the orthocenter and circumcenter of of an acute-angled triangke ABC with $AB < AC$.Let K be the midpoint of AH.The line through K perpendicular to OK meet AB and the tangent to the circumcircle at A at X and Y respectively. Prove that $\angle XOY = \angle AOB$

5 Grade8 P5 of Sharygin 2019 Finals

6 A point H lies on the side AB of regular polygon $ABCDE$. A circle with center H and radius HE meets the segments DE and CD at points G and F respectively. It is known that $DG = AH$. Prove that $CF = AH$.

7 Let points M and N lie on sides AB and BC of triangle ABC in such a way that $MN||AC$. Points M' and N' are the reflections of M and N about BC and AB respectively. Let $M'A$ meet BC at X, and let $N'C$ meet AB at Y. Prove that A, C, X, Y are concyclic.

8 What is the least positive integer k such that, in every convex 1001-gon, the sum of any k diagonals is greater than or equal to the sum of the remaining diagonals?

– Grade 9

1 A triangle OAB with $\angle A = 90^\circ$ lies inside another triangle with vertex O. The altitude of OAB from A until it meets the side of angle O at M. The distances from M and B to the second side of angle O are 2 and 1 respectively. Find the length of OA.

2 Let P be a point on the circumcircle of triangle ABC. Let A_1 be the reflection of the orthocenter of triangle PBC about the reflection of the perpendicular bisector of BC. Points B_1 and C_1 are defined similarly. Prove that A_1, B_1, C_1 are collinear.

3 Let $ABCD$ be a cyclic quadrilateral such that $AD = BD = AC$. A point P moves along the circumcircle ω of triangle $ABCD$. The lined AP and DP meet the lines CD and AB at points E and F respectively. The lines BE and CF meet point Q. Find the locus of Q.

4 A ship tries to land in the fog. The crew does not know the direction to the land. They see a lighthouse on a little island, and they understand that the distance to the lighthouse does not exceed 10 km (the exact distance is not known). The distance from the lighthouse to the land equals 10 km. The lighthouse is surrounded by reefs, hence the ship cannot approach it. Can the ship land having sailed the distance not greater than 75 km?
(*The waterside is a straight line, the trajectory has to be given before the beginning of the motion, after that the autopilot navigates the ship*.)

5 Let R be the circumradius of a circumscribed quadrilateral $ABCD$. Let h_1 and h_2 be the altitudes from A to BC and CD respectively. Similarly h_3 and h_4 are the altitudes from C to AB and AD. Prove that

$$\frac{h_1+h_2-2R}{h_1h_2}=\frac{h_3+h_4-2R}{h_3h_4}$$

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6 A non-convex polygon has the property that every three consecutive its vertices from a right-angled triangle. Is it true that this polygon has always an angle equal to 90° or to 270°?

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7 Let the incircle ω of $\triangle ABC$ touch AC and AB at points E and F respectively. Points X, Y of ω are such that $\angle BXC=\angle BYC=90^\circ$. Prove that EF and XY meet on the medial line of ABC.

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8 A hexagon $A_1A_2A_3A_4A_5A_6$ has no four concyclic vertices, and its diagonals A_1A_4, A_2A_5 and A_3A_6 concur. Let l_i be the radical axis of circles $A_iA_{i+1}A_{i-2}$ and $A_iA_{i-1}A_{i+2}$ (the points A_i and A_{i+6} coincide). Prove that $l_i, i=1,\cdots,6$, concur.

– Grade 10

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1 Given a triangle ABC with $\angle A=45^\circ$. Let A' be the antipode of A in the circumcircle of ABC. Points E and F on segments AB and AC respectively are such that $A'B=BE$, $A'C=CF$. Let K be the second intersection of circumcircles of triangles AEF and ABC. Prove that EF bisects $A'K$.

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2 Let A_1, B_1, C_1 be the midpoints of sides BC, AC and AB of triangle ABC, AK be the altitude from A, and L be the tangency point of the incircle γ with BC. Let the circumcircles of triangles LKB_1 and A_1LC_1 meet B_1C_1 for the second time at points X and Y respectively, and γ meet this line at points Z and T. Prove that $XZ=YT$.

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3 Let P and Q be isogonal conjugates inside triangle ABC. Let ω be the circumcircle of ABC. Let A_1 be a point on arc BC of ω satisfying $\angle BA_1P=\angle CA_1Q$. Points B_1 and C_1 are defined similarly. Prove that AA_1, BB_1, CC_1 are concurrent.

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4 Prove that the sum of two nagelians is greater than the semiperimeter of a triangle. (The nagelian is the segment between the vertex of a triangle and the tangency point of the opposite side with the correspondent excircle.)

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5 Let AA_1, BB_1, CC_1 be the altitudes of triangle ABC, and $A0, C0$ be the common points of the circumcircle of triangle A_1BC_1 with the lines A_1B_1 and C_1B_1 respectively. Prove that AA_0 and CC_0 meet on the median of ABC or are parallel to it

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6 Let AK and AT be the bisector and the median of an acute-angled triangle ABC with $AC>AB$. The line AT meets the circumcircle of ABC at point D. Point F is the reflection of K

about T. If the angles of ABC are known, find the value of angle FDA.

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7 Let P be an arbitrary point on side BC of triangle ABC. Let K be the incenter of triangle PAB. Let the incircle of triangle PAC touch BC at F. Point G on CK is such that $FG//PK$. Find the locus of G.

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8 Several points and planes are given in the space. It is known that for any two of given points there exactly two planes containing them, and each given plane contains at least four of given points. Is it true that all given points are collinear?
