Olympic Revenge 2019

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1 Let ABC be a scalene acute-angled triangle and D be the point on its circumcircle such that AD is a symmedian of triangle ABC. Let E be the reflection of D about BC, C_0 the reflection of E about AB and B_0 the reflection of E about AC. Prove that the lines AD, BB_0 and CC_0 are concurrent if and only if $\angle BAC = 60^\circ$.

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2 Prove that there exist infinitely many positive integers n such that the greatest prime divisor of n^2+1 is less than $n \cdot \pi^{-2019}$.

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3 Let Γ be a circle centered at O with radius R. Let X and Y be points on Γ such that $XY < R$. Let I be a point such that $IX = IY$ and $XY = OI$. Describe how to construct with ruler and compass a triangle which has circumcircle Γ, incenter I and Euler line OX. Prove that this triangle is unique.

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4 A regular icosahedron is a regular solid of 20 faces, each in the form of an equilateral triangle, with 12 vertices, so that each vertex is in 5 edges.
Twelve indistinguishable candies are glued to the vertices of a regular icosahedron (one at each vertex), and four of these twelve candies are special. Andr and Lucas want to together create a strategy for the following game:
First, Andr is told which are the four special sweets and he must remove exactly four sweets that are not special from the icosahedron and leave the solid on a table, leaving afterwards without communicating with Lucas.
Later, Sponchi, who wants to prevent Lucas from discovering the special sweets, can pick up the icosahedron from the table and rotate it however he wants.
After Sponchi makes his move, he leaves the room, Lucas enters and he must determine the four special candies out of the eight that remain in the icosahedron.
Determine if there is a strategy for which Lucas can always properly discover the four special sweets.

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5 Define $f:\mathbb{N}\to\mathbb{N}$ by

$$f(n)=\sum \frac{(1+\sum_{i=1}^{n} t_i)!}{(1+t_1)\cdot\prod_{i=1}^{n}(t_i!)}$$

where the sum runs through all n-tuples such that $\sum_{j=1}^{n} j\cdot t_j = n$ and $t_j \geq 0$ for all $1\leq j\leq n$.
Given a prime p greater than 3, prove that

$$\sum_{1\leq i<j<k\leq p-1}\frac{f(i)}{i\cdot j\cdot k} \equiv \sum_{1\leq i<j<k\leq p-1}\frac{2^i}{i\cdot j\cdot k} \pmod{p}.$$
