Regional Competition For Advanced Students 2015

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by RockmanEX3

1 Determine all triples (a, b, c) of positive integers satisfying the conditions

$$\gcd(a, 20) = b$$
$$\gcd(b, 15) = c$$
$$\gcd(a, c) = 5$$

(Richard Henner)

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2 Let x, y, and z be positive real numbers with $x + y + z = 3$. Prove that at least one of the three numbers

$$x(x + y - z)$$
$$y(y + z - x)$$
$$z(z + x - y)$$

is less or equal 1.

(Karl Czakler)

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3 Let $n \geq 3$ be a fixed integer. The numbers $1, 2, 3, \cdots, n$ are written on a board. In every move one chooses two numbers and replaces them by their arithmetic mean. This is done until only a single number remains on the board.

Determine the least integer that can be reached at the end by an appropriate sequence of moves.

(Theresia Eisenkölbl)

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4 Let ABC be an isosceles triangle with $AC = BC$ and $\angle ACB < 60^\circ$. We denote the incenter and circumcenter by I and O, respectively. The circumcircle of triangle BIO intersects the leg BC also at point $D \neq B$.

(a) Prove that the lines AC and DI are parallel.
(b) Prove that the lines OD and IB are mutually perpendicular.

(Walther Janous)
