

Ukraine National Mathematical Olympiad 2018www.artofproblemsolving.com/community/c854137

by Snakes, Mindstormer

– Grade 11

– Day 1

- 1** Find all triples of pairwise distinct positive integers (a, b, c) , which satisfy the following conditions: $2a - 1$ is divisible by b , $2b - 1$ is divisible c and $2c - 1$ is divisible by a .

Proposed by Bohdan Rublyov

- 2** In acute-angled triangle ABC , AH is an altitude and AM is a median. Points X and Y on lines AB and AC respectively are such that $AX = XC$ and $AY = YB$. Prove that the midpoint of XY is equidistant from H and M .

Proposed by Danylo Khilko

- 3** Find all functions $f : [0, +\infty) \mapsto [0, +\infty)$, which for all nonnegative x, y satisfy

$$f(f(x) + f(y)) = xyf(x + y)$$

Proposed by Igor Voronovich

- 4** Two players - Andriy and Olesya play the following game. On a table there is a round cake, which is cut by one of the players into $2n$ ($n > 1$) sectors (pieces), pairwise distinct in weights. Weight of each piece is known to both players. Then they take pieces according to the following rules. At first, Olesya takes 1 piece, then Andriy takes 2 pieces such that the remaining pieces form a sector. Then they in turns take 2 pieces such that after each turn the remaining pieces form a sector. In their last move, one of the players takes one last piece. Each of the players aims to get more of a cake than the opponent. For which n can Olesya cut the cake such that she wins if in her first move she takes the smallest piece?

Proposed by Bohdan Rublyov

– Day 2

- 5** The squadron of 10 powerful destroyers and 20 small boats is about to attack the island. All ships are positioned on the straight line, and are equally spaced.

Two torpedo boats with 10 torpedoes each want to protect the island. However, the first torpedo boat can shoot only 10 successive boats, whereas the second one can shoot 10 targets

which are next by one. Note that they have to shoot at the same moment, so that some targets may be hit by both torpedoes.

What is the biggest number of destroyers that can avoid the torpedoes no matter which targets the torpedo boats choose?

Proposed by Bohdan Rublyov

-
- 6 The sequence (x_n) is given by $x_1 = a$, $x_{n+1} = \frac{1}{2} \left(x_n - \frac{1}{x_n} \right)$. Prove that there is a such that the sequence (x_n) has exactly 2018 pairwise distinct elements. [i](If some element of the sequence is equal to 0, it stops on that element)[/i]

Proposed by Andriy Hoholev

-
- 7 Given N positive integers such that the greatest common divisors of all nonempty subsets of them are pairwise distinct. What is the smallest number of prime factors of the product of all N numbers?

Proposed by Aleksandr Golovanov

-
- 8 Given an acute-angled triangle ABC , AA_1 and CC_1 are its angle bisectors, I is its incenter, M and N are the midpoints of AI and CI . Points K and L in the interior of triangles AC_1I and CA_1I respectively are such that $\angle AKI = \angle CLI = \angle AIC$, $\angle AKM = \angle ICA$, $\angle CLN = \angle IAC$. Prove that the circumradii of triangles KIL and ABC are equal.

Proposed by Anton Trygub
