Belarusian National Olympiad 2018

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Category C Category C

9.1 Prove that the set of all divisors of a positive integer which is not a perfect square can be divided into pairs so that in each pair one number is divisible by another.

9.2 For every integer $n \geqslant 2$ prove the inequality

$$\frac{1}{2!}+\frac{2}{3!}+\ldots+\frac{2^{n-2}}{n!}\leqslant\frac{3}{2},$$

where $k!=1\cdot 2\cdot\ldots\cdot k$.

9.3 The bisector of angle CAB of triangle ABC intersects the side CB at L. The point D is the foot of the perpendicular from C to AL and the point E is the foot of perpendicular from L to AB. The lines CB and DE meet at F.
Prove that AF is an altitude of triangle ABC.

9.4 Three $n\times n$ squares form the figure Φ on the checkered plane as shown on the picture. (Neighboring squares are tpuching along the segment of length $n-1$.)
Find all $n>1$ for which the figure Φ can be covered with tiles 1×3 and 3×1 without overlapping.`https://pp.userapi.com/c850332/v850332712/115884/DKxvALE-sAc.jpg`

9.5 The quadrilateral $ABCD$ is inscribed in the parabola $y=x^2$. It is known that angle $BAD=90$, the dioganal AC is parallel to the axis Ox and AC is the bisector of the angle BAD.
Find the area of the quadrilateral $ABCD$ if the length of the dioganal BD is equal to p.

9.6 For all positive integers m and n prove the inequality

$$|n\sqrt{n^2+1}-m|\geqslant\sqrt{2}-1.$$

9.7 A point O is choosen inside a triangle ABC so that the length of segments OA, OB and OC are equal to 15,12 and 20, respectively. It is known that the feet of the perpendiculars from O to the sides of the triangle ABC are the vertices of an equilateral triangle.
Find the value of the angle BAC.

9.8 A positive integer n is fixed. Numbers 0 and 1 are placed in all cells (exactly one number in any cell) of a $k\times n$ table (k is a number of the rows in the table, n is the number of the columns

in it). We call a table nice if the following property is fulfilled: for any partition of the set of the rows of the table into two nonempty subsets $R1$ and $R2$ there exists a nonempty set S of the columns such that on the intersection of any row from $R1$ with the columns from S there are even number of $1's$ while on the intersection of any row from $R2$ with the columns from S there are odd number of $1's$.
Find the greatest number of k such that there exists at least one nice $k \times n$ table.

Category B Category B

10.1 The extension of the median AM of the triangle ABC intersects its circumcircle at D. The circumcircle of triangle CMD intersects the line AC at C and E.The circumcircle of triangle AME intersects the line AB at A and F. Prove that CF is the altitude of triangle ABC.

10.2 Determine, whether there exist a function f defined on the set of all positive real numbers and taking positive values such that $f(x+y) \geqslant yf(x) + f(f(x))$ for all positive x and y?

10.3 For a fixed integer $n \geqslant 2$ consider the sequence $a_k = \text{lcm}(k, k+1, \ldots, k+(n-1))$. Find all n for which the sequence a_k increases starting from some number.

10.4 Some cells of a checkered plane are marked so that figure A formed by marked cells satisfies the following condition:$1)$ any cell of the figure A has exactly two adjacent cells of A; and $2)$ the figure A can be divided into isosceles trapezoids of area 2 with vertices at the grid nodes (and acute angles of trapezoids are equal to 45) . Prove that the number of marked cells is divisible by 8.

10.5 Find all positive integers n such that equation

$$3a^2 - b^2 = 2018^n$$

has a solution in integers a and b.

10.6 The vertices of the convex quadrilateral $ABCD$ lie on the parabola $y = x^2$. It is known that $ABCD$ is cyclic and AC is a diameter of its circumcircle. Let M and N be the midpoints of the diagonals of AC and BD respectively. Find the length of the projection of the segment MN on the axis Oy.

10.7 The square $A_1B_1C_1D_1$ is inscribed in the right triangle ABC (with $C = 90$) so that points A_1, B_1 lie on the legs CB and CA respectively ,and points C_1, D_1 lie on the hypotenuse AB. The circumcircle of triangles B_1A_1C an AC_1B_1 intersect at B_1 and Y. Prove that the lines A_1X and B_1Y meet on the hypotenuse AB.

10.8 The vertices of the regular n-gon and its center are marked. Two players play the following game: they, in turn, select a vertex and connect it by a segment to either the adjacent vertex or the center. The winner I a player if after his maveit is possible to get any marked point from any other moving along the segments. For each $n > 2$ determine who has a winning strategy.

Category A Category A

11.1 Find all real numbers a for which there exists a function f defined on the set of all real numbers which takes as its values all real numbers exactly once and satisfies the equality

$$f(f(x)) = x^2 f(x) + ax^2$$

for all real x.

11.2 The altitudes AA_1, BB_1 and CC_1 are drawn in the acute triangle ABC. The bisector of the angle AA_1C intersects the segments CC_1 and CA at E and D respectively. The bisector of the angle AA_1B intersects the segments BB_1 and BA at F and G respectively. The circumcircles of the triangles FA_1D and EA_1G intersect at A_1 and X.
Prove that $\angle BXC = 90^\circ$.

11.3 For all pairs (m, n) of positive integers that have the same number k of divisors we define the operation $\circ$. Write all their divisors in an ascending order: $1 = m_1 < \ldots < m_k = m$, $1 = n_1 < \ldots < n_k = n$ and set

$$m \circ n = m_1 \cdot n_1 + \ldots + m_k \cdot n_k.$$

Find all pairs of numbers (m, n), $m \geqslant n$, such that $m \circ n = 497$.

11.4 A checkered polygon A is drawn on the checkered plane. We call a cell of A *internal* if all 8 of its adjacent cells belong to A. All other (non-internal) cells of A we call *boundary*. It is known that $1)$ each boundary cell has exactly two common sides with no boundary cells; and 2) the union of all boundary cells can be divided into isosceles trapezoid of area 2 with vertices at the grid nodes (and acute angles of the trapezoids are equal 45°).
Prove that the area of the polygon A is congruent to 1 modulo 4.

11.5 The circle S_1 intersects the hyperbola $y = \frac{1}{x}$ at four points A, B, C, and D, and the other circle S_2 intersects the same hyperbola at four points A, B, F, and G. It's known that the radii of circles S_1 and S_2 are equal.
Prove that the points C, D, F, and G are the vertices of the parallelogram.

11.6 The point X is marked inside the triangle ABC. The circumcircles of the triangles AXB and AXC intersect the side BC again at D and E respectively. The line DX intersects the side AC at K, and the line EX intersects the side AB at L.
Prove that $LK \parallel BC$.

11.7 Consider the expression $M(n,m) = |n\sqrt{n^2+a} - bm|$, where n and m are arbitrary positive integers and the numbers a and b are fixed, moreover a is an odd positive integer and b is a rational number with an odd denominator of its representation as an irreducible fraction.
Prove that there is
a) no more than a finite number of pairs (n,m) for which $M(n,m)=0$;
b) a positive constant C such that the inequality $M(n,m) \geqslant 0$ holds for all pairs (n,m) with $M(n,m) \neq 0$.

11.8 The vertices of the regular n-gon are marked. Two players play the following game: they, in turn, select a vertex and connect it by a segment to either the adjacent vertex or the center of the n-gon. The winner is a player if after his move it is possible to get any vertex from any other vertex moving along segments.
For each integer $n \geqslant 3$ determine who has a winning strategy.
