Belarus Team Selection Test 2017

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– Test 1

– Find all prime numbers a and b such that

$$20a^3 - b^3 = 1.$$

– An $n \times n$ square table is divided into n^2 unit cells. Some unit segments of the obtained grid (i.e. the side of any unit cell) is colored black so that any unit cell of the given square has exactly one black side. Find
a) the smallest
b) the greatest possible number of black unit segments.

– Let H be the orthocenter of an acute triangle ABC, $AH = 2$, $BH = 12$, $CH = 9$.
Find the area of the triangle ABC.

– Let four parallel lines l_1, l_2, l_3, and l_4 meet the hyperbola $y = 1/x$ at points A_1 and B_1, A_2 and B_2, A_3 and B_3, A_4 and B_4, respectively.
Prove that the areas of the quadrilaterals $A_1A_2A_3A_4$ and $B_1B_2B_3B_4$ are equal.

– Test 2

– Let A and B be two disjoint subsets of positive integers such that —AI=n, IBI= m. It is known that for any k AUB at least one of the two following conditions holds
1) k + 17 A
2) k -31 B
Find all possible values of m/n.

– Find all positive numbers a, b, c, d such that $a + b + c + d = 1$ and

$$\max\left\{\frac{a^2}{b}, \frac{b^2}{a}\right\} \cdot \min\left\{\frac{c^2}{d}, \frac{d^2}{c}\right\} = (\min\{a + b, c + d\})^4.$$

– Let $1 = d_1 < d_2 < \ldots < d_k = n$ be all natural divisors of a natural number n.
Find all possible values of k if $n = d_2d_3 + d_2d_5 + d_3d_5$.

– Given triangle ABC, let D be an inner point of the side BC. Let P and Q be distinct inner points of the segment AD. Let $K = BP \cap AC$, $L = CP \cap AB$, $E = BQ \cap AC$, $F = CQ \cap AB$.
Given that $KL \parallel EF$, find all possible values of the ratio $BD : DC$.

– Test 3

– Let I be the incenter of a non-isosceles triangle ABC. The line AI intersects the circumcircle of the triangle ABC at A and D. Let M be the middle point of the arc BAC. The line through the point I perpendicular to AD intersects BC at F. The line MI intersects the circle BIC at N.
Prove that the line FN is tangent to the circle BIC.

– Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$4xyf(x^2 - y^2) = (x^2 - y^2)f(2x)f(2y)$$

for all real x and y.

– The following operations are performed to the natural numbers x: $x \to x + 2$ or $x \to x + n$, $x \to x \cdot 2$ or $x \to x \cdot n$. Additions and multiplications are performes alternatively (adding 2 or n and multiplying by 2 or by n one can choose as he wishes for each step). The number m is called *attainable* if it can be obtained from 1 by a sequence of such operations, otherwise m is called *unattainable*.
Prove that if $n = 5$ or $n = 7$, then there are infinitely many unattainable numbers.

– Test 4

– On the side AB of a cyclic quadrilateral $ABCD$ there is a point X such that diagonal BD bisects CX and diagonal AC bisects DX. What is the minimum possible value of $\frac{AB}{CD}$?

Proposed by S. Berlov

– Given that x, y, z are positive real numbers satiafying $x + y + z = 2$, prove the inequality

$$\frac{(x-1)^2}{y} + \frac{(y-1)^2}{z} + \frac{(z-1)^2}{x} \geqslant \frac{1}{4}\left(\frac{x^2+y^2}{x+y} + \frac{y^2+z^2}{y+z} + \frac{z^2+x^2}{z+x}\right).$$

– Prove that for any positive integers a and b there exist infinitely many prime numbers p such that $ap + b$ is a composite number.
(Using Dirichlet's theorem is not allowed.)

– Test 5

– Let X be a finite set. Suppose that $X = A_1 \sqcup \ldots \sqcup A_{10}$ and $X = B_1 \sqcup \ldots \sqcup B_{10}$ are partitions of X satisfying the following condition: if $A_i \cap B_j = \varnothing$ for some $1 \leqslant i \leqslant j \leqslant 10$, then $|A_i \cup B_j| \geqslant 10$. Determine the smallest possible number of the elements in X.

– Let ABC be a triangle with circumcircle Γ and incenter I and let M be the midpoint of $\overline{BC}$. The points D, E, F are selected on sides $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ such that $\overline{ID} \perp \overline{BC}$, $\overline{IE} \perp \overline{AI}$, and $\overline{IF} \perp \overline{AI}$. Suppose that the circumcircle of $\triangle AEF$ intersects Γ at a point X other than A. Prove that lines XD and AM meet on Γ.

Proposed by Evan Chen, Taiwan

– Let a, b, c be the lengths of the sides of a triangle ABC. Prove that

$$a^2(p-a)(p-b) + b^2(p-b)(p-c) + c^2(p-c)(p-a) \leqslant \frac{4}{27}p^4,$$

where p is the half-perimeter of the triangle ABC.

– Test 6

– For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

$$S(P(n)) = P(S(n)).$$

Proposed by Warut Suksompong, Thailand

– Any cell of a 5×5 table is colored black or white.
Find the greatest possible value of the ways to place a T-tetramino on the table so that it covers exactly two white and two black cells (for various colorings of the table).

– Given an isosceles triangle ABC with $AB = AC$. let $\omega(XYZ)$ be the circumcircle of a triangle XYZ. Tangents to $\omega(ABC)$ at B and C meet at D. Point F is marked on the arc AB (opposite to C). Let K, L be the intersection points of AF and BD, AB and CF, respectively.
Prove that if circles $\omega(BTS)$ and $\omega(CFK)$ are tangent to each other, the their tangency point belongs to AB. (Here T and S are the centers of the circles $\omega(BLC)$ and $\omega(BLK)$, respectively.)

– Test 7

– Given positive real a, b, c such that $2abc + ab + bc + ca = 1$, prove the inequality

$$\frac{a+1}{(a+1)^2+(b+1)^2} + \frac{b+1}{(b+1)^2+(c+1)^2} + \frac{c+1}{(c+1)^2+(a+1)^2} \leqslant 1.$$

– Let $\tau(n)$ be the number of positive divisors of n. Let $\tau_1(n)$ be the number of positive divisors of n which have remainders 1 when divided by 3. Find all positive integral values of the fraction $\frac{\tau(10n)}{\tau_1(10n)}$.

– Let n be a positive integer relatively prime to 6. We paint the vertices of a regular n-gon with three colours so that there is an odd number of vertices of each colour. Show that there exists an isosceles triangle whose three vertices are of different colours.

– Test 8

– Find the smallest constant $C > 0$ for which the following statement holds: among any five positive real numbers a_1, a_2, a_3, a_4, a_5 (not necessarily distinct), one can always choose distinct subscripts i, j, k, l such that

$$\left|\frac{a_i}{a_j} - \frac{a_k}{a_l}\right| \le C.$$

– Let ABC be a triangle with $AB = AC \neq BC$ and let I be its incentre. The line BI meets AC at D, and the line through D perpendicular to AC meets AI at E. Prove that the reflection of I in AC lies on the circumcircle of triangle BDE.

– Denote by $\mathbb{N}$ the set of all positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all positive integers m and n, the integer $f(m) + f(n) - mn$ is nonzero and divides $mf(m) + nf(n)$.

Proposed by Dorlir Ahmeti, Albania