Kosovo National Mathematical Olympiad 2019

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by Snakes, dangerousliri, Leartia

–	Grade 9
1	Calculate $1^2-2^2+3^2-4^2+...-2018^2+2019^2$.
2	Show that when the product of three conscutive numbers we add arithmetic mean of them it is a perfect cube.
3	Let $ABCD$ be a rectangle with $AB>BC$. Let points E,F be on side CD such that $CE=ED$ and $BC=CF$. Show that if AC is prependicular to BE then $AB=BF$.
4	Find all sequence of consecutive positive numbers which the sum of them is equal with 2019.
5	There are given in a table numbers $1,2,...,18$. What is minimal number of numbers we should erase such that the sum of every two remaining numbers is not perfect square of a positive integer.
–	Grade 10
1	Find last three digits of the number $\frac{2019!}{2^{1009}}$.
2	Show that for any positive real numbers a,b,c the following inequality is true: $$4(a^3+b^3+c^3+3)\geq 3(a+1)(b+1)(c+1)$$ When does equality hold?
3	The doctor instructed a person to take 48 pills for next 30 days. Every day he take at least 1 pill and at most 6 pills. Show that exist the numbers of conscutive days such that the total numbers of pills he take is equal with 11.
4	Find all real numbers x,y,z such that satisfied the following equalities at same time: $\sqrt{x^3-y}=z-1 \wedge \sqrt{y^3-z}=x-1 \wedge \sqrt{z^3-x}=y-1$
5	Let $ABCDE$ be a regular pentagon. Let point F be intersection of segments AC and BD. Let point G be in segment AD such that $2AD=3AG$. Let point H be the midpoint of side DE. Show that the points F,G,H lie on a line.

– Grade 11

1 Let a, b be real numbers grater then 4. Show that at least one of the trinomials $x^2 + ax + b$ or $x^2 + bx + a$ has two different real zeros.

2 Find all positive integers n such that $6^n + 1$ it has all the same digits when it is writen in decimal representation.

3 Let ABC be a triangle with $\angle CAB = 60^\circ$ and with incenter I. Let points D, E be on sides AC, AB, respectively, such that BD and CE are angle bisectors of angles $\angle ABC$ and $\angle BCA$, respectively. Show that $ID = IE$.

4 Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(xy + f(x)) = xf(y)$$

for all $x, y \in \mathbb{R}$.

5 There are given points with integer coordinate (m, n) such that $1 \leq m, n \leq 4$. Two players, Ana and Ben, are playing a game: First Ana color one of the coordinates with red one, then she pass the turn to Ben who color one of the remaining coordinates with yellow one, then this process they repeate again one after other. The game win the first player who can create a rectangle with same color of vertices and the length of sides are positive integer numbers, otherwise the game is a tie. Does there exist a strategy for any of the player to win the game?

– Grade 12

1 Does there exist a triangle with length a, b, c such that:

a) $\begin{cases} a + b + c = 6 \\ a^2 + b^2 + c^2 = 13 \\ a^3 + b^3 + c^3 = 28 \end{cases}$

b) $\begin{cases} a + b + c = 6 \\ a^2 + b^2 + c^2 = 13 \\ a^3 + b^3 + c^3 = 30 \end{cases}$

2 Suppose that each point on a plane is colored with one of the colors red or yellow. Show that exist a convex pentagon with three right angles and all vertices are with same color.

3 Show that for any non-negative real numbers a, b, c, d such that $a^2 + b^2 + c^2 + d^2 = 1$ the following inequality hold:

$$a + b + c + d - 1 \geq 16abcd$$

When does equality hold?

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4 Let ABC be an acute triagnle with its circumcircle ω. Let point D be the foot of triangle ABC from point A. Let points E,F be midpoints of sides AB,AC, respectively. Let points P and Q be the second intersections of of circle ω with circumcircle of triangles BDE and CDF, respectively. Suppose that A,P,B,Q and C be on a circle in this order. Show that the lines EF,BQ and CP are concurrent.

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5 Find all positive integers x,y such that 2^x+19^y is a perfect cube.
